The three dimensional Kakeya conjecture is true

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The just-published issue of the *Journal of the American Mathematical Society* (JAMS) contains a paper by Wang and Zahl (Wang and Zahl 2026) in which they resolve the so-called *sticky* case of the Kakeya conjecture in $\mathbb{R}^3$. This result became a major milestone on the path toward the full resolution of the conjecture in a subsequent work.

A *Kakeya set* in $\mathbb{R}^3$ is a compact subset of $\mathbb{R}^3$ that contains a unit line segment in every direction. It has long been known that such sets may have Lebesgue measure zero. The Kakeya conjecture in $\mathbb{R}^3$ asserts that every Kakeya set $S \subset \mathbb{R}^3$ must nevertheless have full Hausdorff dimension,

$$\dim_H(S) = 3.$$

The two-dimensional version of the conjecture was proved by Davies, and in particular implies that any Kakeya set $S \subset \mathbb{R}^3$ satisfies $\dim_H(S) \geq 2$. In 1991, Bourgain improved this lower bound to

$$\dim_H(S) \geq \tfrac{7}{3},$$

and in 1995 Wolff further raised it to

$$\dim_H(S) \geq \tfrac{5}{2}.$$

Wolff's bound remained the best known result for more than two decades. In 2019, Katz and Zahl (Katz and Zahl 2019) established the first improvement beyond this threshold, proving that

$$\dim_H(S) \geq \tfrac{5}{2} + \varepsilon_0$$

for some $\varepsilon_0 > 0$. Although quantitatively small, this advance was conceptually significant: Katz and Zahl showed how to rule out certain sets of Hausdorff dimension $5/2$ that closely mimic Kakeya sets, overcoming a key obstruction that had long hindered progress. Their work renewed optimism that a complete solution might be within reach.

In a 2014 lecture, Tao outlined a strategy for resolving the Kakeya conjecture in $\mathbb{R}^3$, now known as the *Katz–Tao program*. A central notion in this program is that of a *sticky Kakeya set*, defined as follows. For each (affine) line $\ell \subset \mathbb{R}^3$, let

$p = p(\ell)$ denote the unique point on ℓ such that the vector from the origin to p is orthogonal to ℓ. The collection $\mathcal{L}$ of all lines in $\mathbb{R}^3$ can be equipped with the metric

$$\rho(\ell, \ell') := |p(\ell) - p(\ell')| + \angle(\ell, \ell'),$$

where $\angle(\ell, \ell')$ denotes the angle between the two lines.

A compact set $S \subset \mathbb{R}^3$ is called a *sticky Kakeya set* if there exists a set of lines $L \subset \mathcal{L}$ with packing dimension 2 that contains at least one line in every direction, and such that for each $\ell \in L$ the intersection $\ell \cap S$ contains a unit line segment. This is precisely the definition of an ordinary Kakeya set, augmented by the requirement that the associated family of lines has packing dimension 2.

The Katz–Tao program proposed a two-step approach:

- (i) Show that any hypothetical counterexample to the Kakeya conjecture in $\mathbb{R}^3$, that is, any Kakeya set of Hausdorff dimension $d < 3$, must be sticky, and also must have two other structural properties called *planiness* and *graininess*.
- (ii) Use these properties to derive a contradiction, thereby ruling out the existence of such counterexamples.

The property of planiness had already been established prior to this program. In 2016, Guth (Guth 2016) proved graininess, leaving stickiness as the final missing ingredient in step (i). In their just-published JAMS paper (Wang and Zahl 2026), Wang and Zahl completed step (ii) of the program by showing that no sticky Kakeya counterexample can exist.

Theorem 1 *Every sticky Kakeya set in* $\mathbb{R}^3$ *has Hausdorff dimension* 3*.*

While paper (Wang and Zahl 2026) was under review, Wang and Zahl posted a subsequent preprint (Wang and Zahl 2025) in which they proved that any Kakeya set in $\mathbb{R}^3$ with Hausdorff dimension $d < 3$ must necessarily be sticky. Combined with Theorem 1, this yields a complete resolution of the Kakeya conjecture in $\mathbb{R}^3$.

Theorem 2 *Every Kakeya set in* $\mathbb{R}^3$ *has Hausdorff dimension* 3*.*

References

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