Almost all Markoff numbers are composite

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Despite the fact that 2026 has only just begun, several striking results with completely elementary statements have already appeared in the 2026 issues of top-tier mathematical journals. A notable example is a paper by Bourgain, Gamburd, and Sarnak (Bourgain, Gamburd, and Sarnak 2026) in the first 2026 issue of the *Journal of the American Mathematical Society*. Among other results, they prove that almost all Markoff numbers are composite and, moreover, that they have have more than v distinct prime factors for any fixed $v \geq 1$.

A *Markoff triple* is a triple of positive integers (x, y, z) satisfying the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz, \tag{1}$$

known as the *Markoff equation*. Any integer that appears as a component of a Markoff triple is called a *Markoff number*.

It is well known, and easy to verify, that all Markoff triples can be generated from the initial solution $(1, 1, 1)$ by repeatedly applying the *Vieta jumping* operations

$$(x, y, z) \mapsto (3yz - x, y, z), \quad (x, y, z) \mapsto (x, 3xz - y, z), \quad (x, y, z) \mapsto (x, y, 3xy-$$

together with permutations of the variables.

The sequence of Markoff triples begins as

$$(1, 1, 1),\ (1, 1, 2),\ (1, 2, 5),\ (1, 5, 13),\ (2, 5, 29),\ (1, 13, 34),\ (1, 34, 89),\ (2, 29, 1$$

while the corresponding sequence of Markoff numbers starts with

$$1, 2, 5, 13, 29, 34, 89, 169, \ldots.$$

Markoff numbers play an important role in several areas of mathematics, including number theory, group theory, and geometry. In particular, they are closely connected to the Lagrange spectrum arising in Diophantine approximation.

Bourgain, Gamburd, and Sarnak proved the following theorem.

Theorem 1 *Almost all Markoff numbers are composite. Moreover, for every fixed $v \geq 1$, we have*

$$|P_v(N)| = o(|M(N)|) \qquad \text{as } N \to \infty,$$

where $M(N)$ denotes the set of all Markoff numbers up to N, and $P_v(N)$ is the set of $m \in M(N)$ that have at most v distinct prime factors.

A key ingredient in the proof is the analysis of solutions to the Markoff equation modulo primes. In 1991, Baragar conjectured that for any prime p and any nonzero solution $(a, b, c) \not\equiv (0, 0, 0)$ of (1) modulo p, there exists an integer solution (x, y, z) of (1) such that

$$(x, y, z) \equiv (a, b, c) \pmod{p}.$$

Bourgain, Gamburd, and Sarnak (Bourgain, Gamburd, and Sarnak 2026) showed that if E denotes the set of primes for which Baragar's conjecture fails, then

$$|\{p \in E \mid p \leq x\}| = O(x^{\varepsilon})$$

for every $\varepsilon > 0$. From this estimate they deduced Theorem 1.

After the appearance of the arXiv version of (Bourgain, Gamburd, and Sarnak 2026), Chen (Chen 2024) proved Baragar's conjecture with at most finitely many exceptions. Let G_p denote the graph whose vertices are solutions to the Markoff equation modulo p, with two vertices joined by an edge if and only if they are related by one of the Vieta jumping operations (taken modulo p). Chen showed that the number of vertices in any connected component of G_p is divisible by p. Combined with the results of (Bourgain, Gamburd, and Sarnak 2026), this implies that G_p is connected for all primes $p > p_0$, where p_0 is an unspecified constant, and hence that Baragar's conjecture holds for all such primes.

Later, Eddy et al. (Eddy et al. 2025) proved that one may take

$$p_0 = 3.45 \times 10^{392}.$$

On the computational side, Brown (Brown 2025) developed in 2025 an almost linear-time algorithm for testing Baragar's conjecture for a given prime p, and used it to verify the conjecture for all primes $p < 10^6$.

Despite this recent progress, many fundamental questions about Markoff numbers remain open. The most famous among them is the *unicity conjecture*, which predicts that for every Markoff number m there exists a unique Markoff triple (x, y, z) such that

$$\max\{x, y, z\} = m.$$

References

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