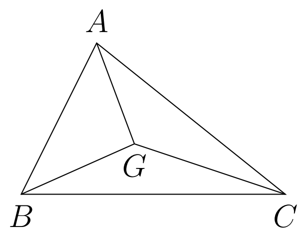
Triangle: Centroid and Area Trisection

Geometricus • 28 Dec 2025

We know that connecting the centroid of a triangle to the vertices gives us three smaller triangles of equal area. That is, if G is the centroid of $\triangle ABC$, then the triangles GBC, GCA and GAB are equal in area.

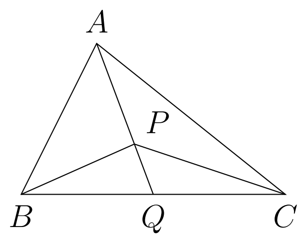


In this article, we are interested in the *converse*:

> If P is a point inside $\triangle ABC$ such that $\triangle GAB$, $\triangle GBC$ and $\triangle GCA$ have equal area, then prove that P must be the centroid.

Proof (I)

Extend AP to meet BC in Q, as shown in the figure below.



$\triangle PBQ$ and $\triangle PCQ$ have a common vertex P and their bases BQ and CQ lie on the same line. Therefore, they have the same height (i.e. distance of P from BC) and their areas are proportional to their bases:

$$\frac{[PBQ]}{[PCQ]} = \frac{BQ}{CQ}$$

$\triangle ABQ$ and $\triangle ACQ$ have a common vertex A and their bases BQ and CQ lie on the same line. Therefore, they have the same height (i.e. distance of A from BC) and their areas are proportional to their bases:

$$\frac{[ABQ]}{[ACQ]} = \frac{BQ}{CQ}$$

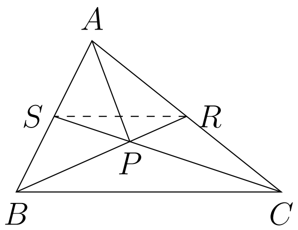
$$\begin{aligned}\therefore \frac{BQ}{CQ} &= \frac{[ABQ]}{[ACQ]} = \frac{[PBQ]}{[PCQ]} \\ &= \frac{[ABQ]-[PBQ]}{[ACQ]-[PCQ]} \quad \ldots \text{(property of equal ratios)} \\ &= \frac{[APB]}{[APC]}\end{aligned}$$

But it is given that $[APB] = [APC]$. Therefore, $BQ = CQ$. This means AQ is the median.

Similarly, BP and CP when extended bisect AC and AB respectively. This proves that P is the centroid of $\triangle ABC$.

Proof (II)

Extend BP to meet AC in R and CP to meet AB in S, as shown in the figure below.



It is given that $[APB] = [BPC] = [APC]$.
$\therefore [APB] = [BPC] = [APC] = \frac{1}{3}[ABC]$

$\triangle APR$ and $\triangle ABR$ have a common vertex A and their bases PR and BR lie on the same line. Therefore, they have the same height and their areas are proportional to their bases:

$$\frac{[APR]}{[ABR]} = \frac{PR}{BR}$$

Similarly,

$$\frac{[CPR]}{[CBR]} = \frac{PR}{BR}$$

$$\begin{aligned}
\therefore \frac{PR}{BR} &= \frac{[APR]}{[ABR]} = \frac{[CPR]}{[CBR]} \\
&= \frac{[APR]+[CPR]}{[ABR]+[CBR]} \\
&= \frac{[APC]}{[ABC]} \\
&= \frac{1}{3} \\
\therefore \frac{PR}{PB} &= \frac{PR}{BR-PR} = \frac{1}{2}
\end{aligned}$$

Similarly,

$$\frac{PS}{PC} = \frac{1}{2}$$

Consider $\triangle SPR$ and $\triangle CPB$:

$$\begin{aligned}
\frac{PS}{PC} &= \frac{PR}{PB} = \frac{1}{2} \\
\angle SPR &= \angle CPB \\
\therefore \triangle SPR &\sim \triangle CPB \\
\therefore \frac{SR}{CB} &= \frac{1}{2} \\
\text{and, } \angle SRP &= \angle CBP \Rightarrow \text{SR is parallel to BC}
\end{aligned}$$

In $\triangle ABC$, SR is parallel to base BC.
$\therefore \triangle ASR \sim \triangle ABC \Rightarrow \frac{AS}{AB} = \frac{SR}{BC} = \frac{1}{2}$. This means that S is the midpoint of AB and CS is a median.

Similarly, BR is also a median. Therefore, P is the centroid of $\triangle ABC$.

Proof (III)

(Proof by contradiction)

Let G be the centroid of $\triangle ABC$.
$\therefore [AGB] = [BGC] = [AGC] = \frac{1}{3}[ABC]$.

Assume that P is a point distinct from the centroid G.

It is given that $[APB] = [BPC] = [APC]$.
$\therefore [APB] = [BPC] = [APC] = \frac{1}{3}[ABC]$

$\triangle BGC$ and $\triangle BPC$ have the same area (equal to $\frac{1}{3}[ABC]$). As they have the same base BC and their respective vertices G and P lie on the same side of line BC, they must have the same height. This means, that points G and P are at the same distance from line BC. Therefore, GP is parallel to BC.

Similarly, we can show that GP is also parallel to AB.

BC intersects AB at B. As GP is parallel to BC, GP must intersect AB.

Now, we have one line GP which is both parallel to AB and intersects AB. This is a contradiction. As our assumption leads to a contradiction, our assumption must be incorrect. Therefore, P must be the centroid of $\triangle ABC$.