Proofs of the Theorems D1–D6 in Chapter 9 of Henry S. Warren, Hacker's Delight, 2nd Edition

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The floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ and the ceiling function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ are defined by

$$\lfloor x \rfloor \equiv \max\{n \in \mathbb{Z} : n <= x\} \text{ and}$$
$$\lceil x \rceil \equiv \min\{n \in \mathbb{Z} : n >= x\},$$

respectively.

Theorem D1. *For x real, k an integer:*

(a) $\lfloor x \rfloor = -\lceil -x \rceil$;

(b) $\lceil x \rceil = -\lfloor -x \rfloor$;

(c) $x - 1 < \lfloor x \rfloor \leq x$;

(d) $x \leq \lceil x \rceil < x + 1$;

(e) $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$;

(f) $\lceil x \rceil - 1 < x \leq \lceil x \rceil$;

(g) $x \geq k \Leftrightarrow \lfloor x \rfloor \geq k$;

(h) $x \leq k \Leftrightarrow \lceil x \rceil \leq k$;

(i) $x > k \Rightarrow \lfloor x \rfloor \geq k$;

(j) $x < k \Rightarrow \lceil x \rceil \leq k$;

(k) $x \leq k \Rightarrow \lfloor x \rfloor \leq k \Rightarrow x < k + 1$;

(l) $x \geq k \Rightarrow \lceil x \rceil \geq k \Rightarrow x > k - 1$;

(m) $x < k \Leftrightarrow \lfloor x \rfloor < k$; *and*

(n) $x > k \Leftrightarrow \lceil x \rceil > k$.

Proof. By the from the difinitions of the floor and ceiling functions, we have that

$$\begin{aligned}\lfloor x \rfloor &= \max\{n \in \mathbb{Z} : n <= x\} = -\min\{-n \in \mathbb{Z} : n <= x\} \\ &= -\min\{-n \in \mathbb{Z} : -n >= -x\} = -\min\{m \in \mathbb{Z} : m >= -x\} \\ &= -\lceil -x \rceil.\end{aligned}$$

This establishes claim (a). Claim (b) is true, because

$$\lceil x \rceil = -(-\lceil -(-x) \rceil) = -\lfloor -x \rfloor,$$

where the second equality follows from the first claim (a).

In the interval $(x - 1, x]$, there is one and only one integer. Because it is the largest among all integers below or equal to x, it must be $\lfloor x \rfloor$. Claim (c) therefore follows. We can verify claim (d) by applying claim (c) setting x to $-x$ and applying claim (b).

The first inequlity in claim (e) is true by the second inequlity in claim (c). Suppose that the second inequality in claim (e) is violated. Then the integer $\lfloor x \rfloor + 1$ is no greater than x but it is greater than $\lfloor x \rfloor$; a contradiction. To prove claim (f), we replace x in claim (e) with $-x$ and apply claim (a) to obtain:

$$-\lceil x \rceil \leq -x < -\lceil x \rceil + 1$$

Multiplying each part of this inequality by -1 establishes the desired result.

Claims (g)–(n) are direct consequences of the definitions of the floor and ceiling functions, except for the second implications in claims (k) and (l). To verify the second implication in claim (k), we establish its contraposition. Suppose that $x \geq k+1$. Then $k+1$ is an integer no greater than x, so that $\lfloor x \rfloor \geq k+1 > k$. The second implication in claim (l) can be verified analogously. □

Theorem D2. *For n, d integers, $d > 0$,*

$$\left\lfloor \frac{n}{d} \right\rfloor = \left\lceil \frac{n-d+1}{d} \right\rceil \tag{1}$$

and

$$\left\lceil \frac{n}{d} \right\rceil = \left\lfloor \frac{n+d-1}{d} \right\rfloor . \tag{2}$$

If $d < 0$:

$$\left\lfloor \frac{n}{d} \right\rfloor = \left\lceil \frac{n-d-1}{d} \right\rceil \tag{3}$$

and

$$\left\lceil \frac{n}{d} \right\rceil = \left\lfloor \frac{n+d+1}{d} \right\rfloor . \tag{4}$$

Proof. Application of Theorem D1 (c) and (d) yields that

$$\frac{n}{d} - 1 < \left\lfloor \frac{n}{d} \right\rfloor \leq \frac{n}{d}$$

and

$$\frac{n-d+1}{d} \leq \left\lceil \frac{n-d+1}{d} \right\rceil < \frac{n-d+1}{d} + 1$$

It follows that we have that

$$-\frac{1}{d} < \left\lfloor \frac{n}{d} \right\rfloor - \left\lceil \frac{n-d+1}{d} \right\rceil \leq 1 - \frac{1}{d}.$$

Becuase the middle term in this inequality is integer, it must be zero. This establishes (1). We can obtain (2) by setting n to $n+d-1$ in (1). To verify (3) and (4), apply (1) and (2), respectively, with the fact that $n/d = (-n)/(-d)$. □

Theorem D3. *For x real, d an integer> 0:*

$$\lfloor \lfloor x \rfloor / d \rfloor = \lfloor x/d \rfloor \text{ and} \tag{5}$$

$$\lceil \lceil x \rceil / d \rceil = \lceil x/d \rceil . \tag{6}$$

Proof. Let $a \equiv \lfloor x/d \rfloor$, $b \equiv \lfloor x - ad \rfloor$, and $c \equiv x - ad - b$. Then it holds that $0 \leq b \leq d-1$, $0 \leq c < 1$, and $0 \leq b + c < d$. It follows that $\lfloor x \rfloor = ad + b$, and that

$$\left\lfloor \frac{\lfloor x \rfloor}{d} \right\rfloor = \left\lfloor \frac{ad+b}{d} \right\rfloor = \left\lfloor a + \frac{b}{d} \right\rfloor = a,$$

where the last equality holds because a is integer, and $0 \leq b/d < 1$. We also have that

$$\left\lfloor \frac{x}{d} \right\rfloor = \left\lfloor \frac{ad+b+c}{d} \right\rfloor = \left\lfloor a + \frac{b+c}{d} \right\rfloor = a,$$

because $0 \leq (b+c)/d < 1$. The equality (5) therefor follows.

For (6), application of Theorem D1(b) yields that

$$\left\lceil \frac{\lceil x \rceil}{d} \right\rceil = \left\lceil -\frac{\lfloor -x \rfloor}{d} \right\rceil = -\left\lfloor \frac{\lfloor -x \rfloor}{d} \right\rfloor$$

and

$$\left\lceil \frac{x}{d} \right\rceil = -\left\lfloor \frac{-x}{d} \right\rfloor.$$

By (5), the right-hand side of these equalities are equal so that (6) holds. □

Corollary. *For a, b real, $b \neq 0$, d an integer> 0,*

$$\left\lfloor \left\lfloor \frac{a}{b} \right\rfloor /d \right\rfloor = \left\lfloor \frac{a}{bd} \right\rfloor$$

and

$$\left\lceil \left\lceil \frac{a}{b} \right\rceil /d \right\rceil = \left\lceil \frac{a}{bd} \right\rceil$$

Proof. Setting x to a/b in Theorem D3 yields the desired results. □

Theorem D4. *Let n and d be integers such tat $d \neq 0$, and x real.*

(a) If $0 \leq x < |1/d|$, then

$$\left\lfloor \frac{n}{d} + x \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor; \text{ and}$$

(b) If $-|1/d| < x \leq 0$,

$$\left\lceil \frac{n}{d} + x \right\rceil = \left\lceil \frac{n}{d} \right\rceil.$$

Proof. Write $\tilde{n} \equiv \text{sgn}(d) \cdot n$ Then $n/d = \tilde{n}/|d|$. Let $a \equiv \lfloor n/d \rfloor = \lfloor \tilde{n}/|d| \rfloor$ and $b \equiv \tilde{n} - a|d|$. Then we have that $0 \leq b \leq |d| - 1$. Under the assumption of claim (a), we also have that $0 \leq b + |d|x < b + 1 \leq |d|$. It thus holds that

$$\left\lfloor \frac{n}{d} + x \right\rfloor = \left\lfloor \frac{\tilde{n}}{|d|} + x \right\rfloor = \left\lfloor \frac{a|d| + b + |d|x}{|d|} \right\rfloor = \left\lfloor a + \frac{b + |d|x}{|d|} \right\rfloor = a = \left\lfloor \frac{n}{d} \right\rfloor.$$

Claim (a) therefore follows.

To verify claim (b), we apply Theorem D1(b) and claim (a) of the current theorem along with the fact that $0 \leq -x < |1/d|$ under the assumption of claim (b):

$$\left\lceil \frac{n}{d} + x \right\rceil = \left\lceil -\left(-\frac{n}{d} - x\right) \right\rceil = -\left\lfloor \frac{-n}{d} + (-x) \right\rfloor = -\left\lfloor \frac{-n}{d} \right\rfloor = \left\lceil \frac{n}{d} \right\rceil.$$

□

The truncating division operator $(x, y) \mapsto x \div y : \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \to \mathbb{R}$ is defined by

$$x \div y \equiv \begin{cases} \max\{z \in \mathbb{Z} : \ zy \leq x\}, \text{ if } x \geq 0, \\ \min\{z \in \mathbb{Z} : \ zy \geq x\}, \text{ otherwise.} \end{cases} \tag{7}$$

The remainder function $\text{rem} : \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \to \mathbb{R}$ is defined by

$$\begin{aligned} \text{rem}(x, y) &\equiv x - x \div y \cdot y \\ &= \begin{cases} \min\{x - zy : \ z \in \mathbb{Z}, \ zy \leq x\}, \text{ if } x \geq 0, \\ -\min\{zy - x : \ z \in \mathbb{Z}, \ zy \geq x\}, \text{ otherwise.} \end{cases} \end{aligned} \tag{8}$$

Theorem D5. *For integers $n \geq 0$, $d \neq 0$,*

$$\text{rem}(2n, d) = \begin{cases} 2\,\text{rem}(n, d), & \text{if } 2\,\text{rem}(n, d) \leq |d| - 1; \\ 2\,\text{rem}(n, d) - |d|, & \text{otherwise.} \end{cases} \tag{9}$$

and

$$\text{rem}(2n + 1, d) = \begin{cases} 2\,\text{rem}(n, d) + 1, & \text{if } 2\,\text{rem}(n, d) \leq |d| - 2; \\ 2\,\text{rem}(n, d) - |d| + 1, & \text{otherwise.} \end{cases} \tag{10}$$

Note: This next theorem has been modified for clarification.

Proof. Let $a \equiv n \div d$ and $b \equiv \text{rem}(n, d)$. Then we have that $0 \leq b \leq |d| - 1$ and that

$$2n = 2(ad + b) = 2ad + 2b$$

If $2b \leq |d| - 1$, we see that $2n \div d = 2a$ and $\text{rem}(2n, d) = 2b$. Otherwise, it holds that

$$|d| \leq 2b \leq 2(|d| - 1) = |d| + (|d| - 2).$$

It follows that $2n \div d = 2a + 1$ and $\text{rem}(2n, d) = 2b - |d|$. The equality (9) therefore follows.

For (10), we have that

$$2n + 1 = 2ad + 2b + 1,$$

where $0 \leq 2b + 1 \leq 2|d| - 1$. If $2b \leq |d| - 2$ (i.e., $2b + 1 \leq |d| - 1$), it holds that $(2n + 1) \div d = 2a$ and $\text{rem}(2n + 1, d) = 2b + 1$; otherwise, it holds that

$$|d| \leq 2b + 1 \leq |d| + (|d| - 1),$$

so that $(2n + 1) \div d = 2a + 1$ and $\text{rem}(2n + 1, d) = 2b + 1 - |d|$, as (10) claims. □

Theorem D6. *For integers $n \geq 0$ and $d \neq 0$,*

$$\text{rem}(2n, 2d) = 2\,\text{rem}(n, d).$$

Proof. Suppose that $n \geq 0$. Then it follows from (8) that

$$\begin{aligned} \operatorname{rem}(2d, 2n) &= \min\{2n - z2d :\ z \in \mathbb{Z},\ z(2d) \leq 2n\} \\ &= \min\{2(n - zd) :\ z \in \mathbb{Z},\ zd \leq n\} \\ &= 2\min\{n - zd :\ z \in \mathbb{Z},\ zd \leq n\} \\ = 2\operatorname{rem}(d, n). \end{aligned}$$

Thus, the equality in question holds. We can also analogously verify the equality for the case in which $n < 0$. □