# Extreme Value Theorem: An Intuitive Approach

Nathan • 26 Feb 2026

## Theorem

A continuous function $f$ defined on interval $[a, b]$ takes on an absolute minimum and absolute maximum value: for some $x_0, x_1 \in [a, b]$ and for all $x \in [a, b]$

$$f(x_0) \leq f(x) \leq f(x_1)$$

## Proof

**Part 1: Absolute Maximum**

We first focus on the maximum of $f$. To do this, we focus on the part right before we hit the max and rely on continuity to show that the maximum must have a preimage under $f$.

We now construct "windows" into our function. Since $f([a, a]) = f(a) \in f([a, b])$, $f([a, b])$ nonempty. Since $f$ on $[a, b]$ and continuous, $f([a, b])$ bounded (continuity pulls bound through from $a$ to $b$). Hence for all $a \leq x \leq b$, $\sup f([a, x])$ exists. In particular, let $K = \sup f([a, b])$. Since there is no asymptotic behavior and $\sup f([a, b])$ is the least upper bound, it would be plausible to conclude (1) $K$ is the maximum value, and (2) it is the image of some element of $x$. Since we are working in a closed region, we shouldn't be able to have function values infinitely approach a point but not reach it.

Now, we formalize the window. Let

$$X = \{x \in [a, b] : \sup f([a, x]) < K\}$$

It seems plausible that the supremum of $X$ is the precise point at which this transition happens and where the function finally reaches the max point.

Assume $f(a) < K$ (if $f(a) = K$, we have found our desired $x$). Then $f(a) = f([a, a]) = \sup f([a, a]) \in X$. Also, $X \subseteq [a, b]$ so since $X$ bounded above and nonempty, exists $c = \sup X$.

If $f(c) = K$, we are done (and which should happen). Otherwise, let $f(c) < K$.

Next, due to continuity, exists a region of inputs centered at $c$ whose outputs are strictly less than $K$. Because $c = \sup X$, this neighborhood must overlap with $X$. By choosing an element within this overlap, we can pull that bound entirely through $c$ (and possibly slightly to the right). Since this structural overlap exists, $c$ cannot be the boundary where the function reaches its supremum, a contradiction.

Formally, let $K - f(c) > \epsilon > 0$. By continuity, exists $\delta > 0$ s.t. $x \in [a, b]$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. We show an immediate consequence that allows us to glue together the bounds: Since $c = \sup X$, $c - \delta < x_0$ for some $x_0 \in X$. Thus by continuity, since for $x_0 \leq t \leq c$, $|c - t| < \delta$ so $|f(t) - f(c)| < \epsilon$. Hence $f(t) = f(t) - f(c) + f(c) < \epsilon + f(c)$ so $\sup f([x_0, c]) \leq \epsilon + f(c) < K$. Since $x_0 \in X$, $\sup f([a, x_0]) < K$ so $\sup f([a, c]) < K$, and so $c \in X$.

If $c = b$, then $b = c \in X$ so $\sup f([a, b]) < K$ contradiction.

If $c < b$, we have some wiggle-room between $c$, our supposed upper bound, and $b$. Let $b - c = \beta > 0$. Consider $x_0 > c$ s.t. $x_0 - c < \min(\delta, \beta)$. Then by continuity, $|f(t) - f(c)| < \epsilon$ so $f(t) = f(t) - f(c) + f(c) < \epsilon + f(c)$ for $c \leq t \leq x_0$. Hence $\sup f([c, x_0]) \leq \epsilon + f(c) < K$. Since $\sup f([a, c]) < K$, $\sup f([a, x_0]) < K$ and $x_0 \in X$ while $x_0 > c$, contradiction as $c$ upper bound.

Hence $f(c) = K = \sup f([a, c])$, and we have found an $x \in [a, c]$ that takes on the absolute maximum value.

**Part 2: Absolute Minimum**

Showing the existence of $x \in [a, b]$ that takes on an absolute minimum value can be done similarly.