Extreme Value Theorem: An Intuitive Approach

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Theorem

A continuous function f defined on interval $[a, b]$ takes on an absolute minimum and absolute maximum value: for some $x_0, x_1 \in [a, b]$ and for all $x \in [a, b]$

$$f(x_0) \leq f(x) \leq f(x_1)$$

Proof

Part 1: Absolute Maximum

We first focus on the maximum of f. To do this, we focus on the part right before we hit the max and rely on continuity to show that the maximum must have a preimage under f.

We now construct "windows" into our function. Since $f([a, a]) = f(a) \in f([a, b])$, $f([a, b])$ nonempty. Since f on $[a, b]$ and continuous, $f([a, b])$ bounded (continuity pulls bound through from a to b). Hence for all $a \leq x \leq b$, $\sup f([a, x])$ exists. In particular, let $K = \sup f([a, b])$. Since there is no asymptotic behavior and $\sup f([a, b])$ is the least upper bound, it would be plausible to conclude (1) K is the maximum value, and (2) it is the image of some element of x. Since we are working in a closed region, we shouldn't be able to have function values infinitely approach a point but not reach it.

Now, we formalize the window. Let

$$X = \{x \in [a, b] : \sup f([a, x]) < K\}$$

It seems plausible that the supremum of X is the precise point at which this transition happens and where the function finally reaches the max point.

Assume $f(a) < K$ (if $f(a) = K$, we have found our desired x). Then $f(a) = f([a, a]) = \sup f([a, a]) \in X$. Also, $X \subseteq [a, b]$ so since X bounded above and nonempty, exists $c = \sup X$.

If $f(c) = K$, we are done (and which should happen). Otherwise, let $f(c) < K$.

Next, due to continuity, exists a region of inputs centered at c whose outputs are strictly less than K. Because $c = \sup X$, this neighborhood must overlap with X. By choosing an element within this overlap, we can pull that bound entirely through c (and possibly slightly to the right). Since this structural overlap exists, c cannot be the boundary where the function reaches its supremum, a contradiction.

Formally, let $K - f(c) > \epsilon > 0$. By continuity, exists $\delta > 0$ s.t. $x \in [a, b]$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. We show an immediate consequence that allows us to glue together the bounds: Since $c = \sup X$, $c - \delta < x_0$ for some $x_0 \in X$. Thus by continuity, since for $x_0 \leq t \leq c$, $|c - t| < \delta$ so $|f(t) - f(c)| < \epsilon$. Hence $f(t) = f(t) - f(c) + f(c) < \epsilon + f(c)$ so $\sup f([x_0, c]) \leq \epsilon + f(c) < K$. Since $x_0 \in X$, $\sup f([a, x_0]) < K$ so $\sup f([a, c]) < K$, and so $c \in X$.

If $c = b$, then $b = c \in X$ so $\sup f([a, b]) < K$ contradiction.

If $c < b$, we have some wiggle-room between c, our supposed upper bound, and b. Let $b - c = \beta > 0$. Consider $x_0 > c$ s.t. $x_0 - c < \min(\delta, \beta)$. Then by continuity, $|f(t) - f(c)| < \epsilon$ so $f(t) = f(t) - f(c) + f(c) < \epsilon + f(c)$ for $c \leq t \leq x_0$. Hence $\sup f([c, x_0]) \leq \epsilon + f(c) < K$. Since $\sup f([a, c]) < K$, $\sup f([a, x_0]) < K$ and $x_0 \in X$ while $x_0 > c$, contradiction as c upper bound.

Hence $f(c) = K = \sup f([a, c])$, and we have found an $x \in [a, c]$ that takes on the absolute maximum value.

Part 2: Absolute Minimum

Showing the existence of $x \in [a, b]$ that takes on an absolute minimum value can be done similarly.