Prove sqrt(2) in R by showing x · x = 2 where x = A|B is the cut in Q with A = {r in Q : r ≤ 0 or r² < 2}

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Question: Prove $\sqrt{2} \in \mathbb{R}$ by showing $x \cdot x = 2$ where $x = A|B$ is the cut in $\mathbb{Q}$ with $A = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}$.

Let $x = A|B$ be Dedekind cut with

$$A = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}, \quad B = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 > 2\}$$

Note $x = \text{lub } A$ (this can be proved separately for cut $x = A|B$). Since $1^2 = 1 < 2$, $1 \in A$, $x \geq 0$. We show $x^2 = 2$.

Case $x^2 < 2$. Let $2 - x^2 = \epsilon > 0$. Since $\frac{\epsilon}{2x+1} \in \mathbb{R}^+$, by Archimedes principle, exists $n \in \mathbb{N}$ s.t. $1/n < \frac{\epsilon}{2x+1}$. Then,

$$(x + 1/n)^2 = x^2 + 2x/n + 1/n^2 < x^2 + \frac{2x+1}{n} < x^2 + \epsilon = 2$$

By density of $\mathbb{Q}$, exists q s.t. $x < q < x + 1/n$. So $q^2 < (x + 1/n)^2 < 2$ and thus $q \in A$. Since $x < q$ and $q \in A$, contradiction that x is upper bound of A.

Case $x^2 > 2$. Let $x^2 - 2 = \epsilon > 0$. Since $\frac{\epsilon}{2x+1} \in \mathbb{R}^+$, $1/n < \frac{\epsilon}{2x+1}$ for some $n \in \mathbb{N}$.

$$(x - 1/n)^2 = x^2 - 2x/n - 1/n^2 > x^2 - \frac{2x+1}{n} > x^2 - \epsilon = 2$$

By density of $\mathbb{Q}$, exists $q \in \mathbb{Q}$ s.t. $2 < x - 1/n < q < x$ so $2 < q < x$ and hence $q > a$ for all $a \in A$ and is thus upper bound smaller than supposed lowest upper bound x, contradiction.

Hence $x^2 = 2$ so $x = \sqrt{2}$.