An integer n>=3 can be expressed as a sum of 2 or more positive integers if and only if n is not a power of 2.

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Theorem

An integer $n \geq 3$ can be expressed as a sum of 2 or more positive integers if and only if n is not a power of 2.

Proof

Part 1: Forward Direction

Assume $n \geq 3$ is not a power of 2. We find consecutive integers starting at a with length K such that their sum is n.

Note:

$$(a) + (a+1) + \cdots + (a+K-1) = aK + \frac{K(K-1)}{2} = \frac{K}{2}[2a+K-1]$$

Thus, we find a, K such that $\frac{K}{2}[2a+K-1] = n$ or, identically,

$$K[2a+K-1] = 2n \tag{1}$$

Note that $2a+K-1$ and K have opposite parity (difference is odd), so we must decompose $2n$ (and thus n) into an even and odd part. To do this, we take out successive factors of 2 such that an odd part remains. Hence, $n = 2^r s$ with nonnegative integer r and odd integer $s \geq 3$, which is guaranteed to exist as n is not a power of 2. Let $s = 2t+1$ where $t \in \mathbb{Z}$.

We now decompose $2n = 2^{r+1}s$ by assigning 2^{r+1} and s to K and $2a+K-1$ without introducing negative integers.

Case 1: $s < 2^{r+1}$

Let $K = s$, $2a+K-1 = 2^{r+1}$.

$$a = \frac{2^{r+1} - 2t - 1 + 1}{2} = 2^r - t.$$

$K \in \mathbb{Z}$ and a is a nonnegative integer as $2^{r+1} > s = 2t + 1 > 2t$, so $2^r > t$.

Hence $n = (2^r - t) + (2^r - t + 1) + \cdots + (2^r + t + 1)$, as desired.

Case 2: $s > 2^{r+1}$

Let $K = 2^{r+1}$, $2a + K - 1 = s$.

$$a = \frac{2t + 1 + 1 - 2^{r+1}}{2} = t - 2^r + 1$$

$K \in \mathbb{Z}$ and a is a nonnegative integer as
$2t + 2 > s = 2t + 1 > 2^{r+1} \implies t + 1 > 2^r$.

Hence, $n = (t - 2^r + 1) + (t - 2^r + 2) + \cdots + (t - 2^r + 2^{r+1})$, as desired.

Part 2: Reverse Direction

Assume to the contrary n is a power of 2, $n = 2^r$ for some positive integer r.

Since $2[(a) + (a + 1) + \cdots + (a + K - 1)] = K[2a + K - 1]$ and the two terms K and $2a + K - 1$ have different parity, there must be an odd factor of $2n$ and thus n for it to be expressed as a sum of 2 or more positive integers. As $n = 2^r$, there are no odd factors so this expansion is impossible.