Grasping Ring Spectra I

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During my final year in undergrad, I used to attend a lot of seminar talks given by the arithmetic geometry research groups, mostly as procrastination on my coursework or undergrad thesis. It is safe to say I didn't understand a word they were talking about in those seminars. Most of the times though, I knew that the mathematics presented in those talks was at least somewhat related to something tractable, something concrete.

Until one very particular talk. This talk, I remember, caused in me a very specific feeling of alienation from anything mathematically sensible. I forgot almost completely what it was about, though what stuck to me, very vividly, is that the speaker was constantly throwing around the words *t-structure*, *sphere spectrum*, *stable ∞-category* and many other fearsome words. If it weren't for the blackboard usage, or the excessive amount of hair loss among the (almost fully male) audience, I would not have been able to tell whether I was attending a mathematical talk or a pseudoscientific attempt of a self-proclaimed quantum physicist in proving how aluminum foil protects against 5G-waves, using all technical terms at his disposal. "*What the fuck is this sphere spectrum?*" was my first question after finally being freed from this 90-minute long psychological torment.

Two years later, as destiny would have it, the *sphere spectrum* has become one of the main characters of my mathematical life, as I'm learning some *brave new algebra* for my thesis work on Algebraic K-Theory. Just as those countless talks I attended during my undergrad, this (very first!!) blog post is again a means of procrastination on my thesis work. But more than that, it is also an attempt in answering my very own question: *What the fuck is this sphere spectrum?*

As the name suggests, the sphere spectrum has something to do with spheres. Now since calculating the homotopy groups of spheres seems like an impossible task, we have scaled down our ambitions into computing their *stable* homotopy groups, which are defined as the colimit

$$\pi_n^{\mathrm{st}} := \operatorname{colim}_k \pi_{n+k}(S^k).$$

In fact, by the Freudenthal Suspension Theorem, this colimit stabilises, explaining the terminology. Using the adjunction $\Sigma^k \dashv \Omega^k$, we can rewrite these groups as

$$\pi_n^{\mathrm{st}} = \operatorname{colim}_k \pi_n(\Omega^k \Sigma^k S^0) = \pi_n(\operatorname{colim}_k \Omega^k \Sigma^k S^0).$$

So ultimately, we are just computing the homotopy groups of some ordinary space $\mathrm{colim}_k\, \Omega^k \Sigma^k S^0$! Let's denote this space $Q(S^0)$. By unwinding definitions, we observe that $Q(S^k) \to \Omega Q(S^{k+1})$ is an equivalence, so that the family $(Q(S^k))_k$ assemble into a spectrum $\mathbb{S}$, namely the **sphere spectrum**. You might also notice that in the construction of $Q(S^0)$, we can actually replace S^0 by any other space X as we're only taking loop spaces, suspension and colimits. This yields a functor

$$\Sigma^\infty : \mathbf{An}_* \to \mathbf{Sp}, X \mapsto (Q(\Sigma^k X))_k$$

from pointed spaces (aka anima) to spectra. Viewing the zero sphere S^0 as $(\mathrm{pt})_+$, the *point with an external basepoint*, we can interpret the sphere spectrum $\mathbb{S} = \Sigma^\infty(\mathrm{pt})_+$ as the *free spectrum on a point*. But why is Σ^∞ even the "right" way to produce a spectrum out of a space? The suspension functor Σ^∞ is left adjoint to Ω^∞, the functor sending a spectrum $(X_k)_k$ to its zeroth space X_0. If you think Ω^∞ is the canonical way to associate a space to a spectrum, then Σ^∞ should surely be the canonical way to associate a spectrum to a space.

There are many reasons for why studying $\mathbb{S}$ is interesting: If you want to study a cohomology theory E^* for example, Brown Representability tells you that you can also study its representing spectrum E. Now as a "free" spectrum, $\mathbb{S}$ should contain all kinds of "relations" to build E, just how any abelian group has a presentation by generators and relations in $\mathbb{Z}$. So in studying $\mathbb{S}$, you're basically also studying all suitable cohomology theories (This blogpost elaborates very well on this). But there is way more to $\mathbb{S}$. Just how $\mathbb{Z}$ is the free abelian group on a point, $\mathbb{S}$ is the free spectrum on a point, which is just one of many analogies between Abelian groups and Spectra. The algebraic properties of $\mathbb{S}$ and its analogy to $\mathbb{Z}$ will (hopefully) be the content of a future blogpost :))

Spotted a few days ago