Intermediate Counting and Probability - Chapter 2

crashing and burning • 14 Feb 2026

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\begin{center}
\section*{Chapter 2}
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\begin{enumerate}
    \item[2.1] This problem is split into $5$ parts.
    \begin{enumerate}
        \item Since every element of $A$ is in $A$, we have $A\subseteq A$. However, $A=A$, which means $A\not\subset A$.
        \item Since every element of $B$ is in $A$, we have $B\subseteq A$. Also, $B\neq A$, which means $B\subset A$ as well.
        \item We have an element $\{4,5\}\in C$ which is not in $A$, so we have $C\not\subseteq A$ and $C\not\subset A$.
        \item We have the elements of $B$ are $2$,$3$, and $4$, so $4\in B$. However, the elements of $C$ are $3$ and $\{4,5\}$, so $4\not\in C$.
        \item The subsets of $B$ are $\{\}$, $\{4\}$, $\{3\}$, $\{2\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, and $\{2,3,4\}$. There are $8$ subsets in all.
    \end{enumerate}
    \item[2.2]
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Since `$\emptyset$` has no elements, every element of `$\emptyset$` is in any set, as there is no element to check it for. So, the empty set is a subset of any set, as desired.

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\item[2.3]
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We have `$B$` is the set of all subsets of `$A$` such that `$1$` is not in said subset. From here, it is easy to list out the elements of `$B$`: `$$B=\{\{\}, \{2\},\{3\},\{6\},\{2,3\},\{2,6\},\{3,6\},\{2,3,6\}\}.$$`

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\item[2.2.1]
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We have `$\{\emptyset\}$` is the set with `$\emptyset$` as its only element. Since `$\emptyset$` is the set with no elements, this means `$\emptyset\neq\{\emptyset\}$`.

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\item[2.2.2]
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This problem is split into `$7$` parts.

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\begin{enumerate}
        \item Since $\{1,3,5,9\}$ has elements $1$,
$3$, $5$, and $9$, we have that our statement is true.
        \item Since we can remove the $2$ in the right
hand side (redundancy), and order does not matter, we
have that our statement is true.
        \item Since the element so of $\{3,5,9\}$ are
$3$, $5$, and $9$, and $\{5\}$ is not among these
numbers, we have that our statement is false.
        \item The only element of $\{\{4\}\}$ is $\{4\}
$, which is not the same as $4$. So, our statement is
false.
        \item Since the empty set is a subset of any
set by 2.2, and $\emptyset\neq\{1,2,9\}$, we have that
our statement is true.
        \item We have the elements of $\{\emptyset,
\{1\},82\}$ are $\emptyset$, $\{1\}$, and $82$. Since $
\emptyset$ is included in these elements, we have our
statement is true.
        \item Since every even integer is an integer,
and there are integers which are not integers, we have
$\{x\mid x\textrm{ is an even integer}\}
\subseteq\mathbb{Z}$ and $\{x\mid x\textrm{ is an even
integer}\}\neq\mathbb{Z}$. Combining these, we get that
our statement is true.
    \end{enumerate}
    \item[2.2.3]
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Since every element of `$A$` is in `$A$`, by the definition of `$\subseteq$` we have `$A\subseteq A$`.

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\item[2.2.4]
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`$A\subseteq B$` means every element of `$A$` is in `$B$`. `$B\subseteq A$` means every element of `$B$` is in `$A$`. Suppose, for the sake of contradiction, that we can take an element `$x$` such that `$x\in A$` but `$x\not\in B$`. (The choice of `$A$` and `$B$` can be switched easily.) This is the same as `$A\neq B$`. However, all elements in `$A$` must be in `$B$`, so this `$x$` cannot exist. (The choice of `$x\in B$` and `$x\not\in A$` can be disproved similarly.) Therefore, we have a contradiciton, so `$A=B$`, as desired.

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\item[2.2.5]
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Suppose we have two sets such that `$A\subseteq B$` and `$B\subseteq C$`. This means that every element in `$A$` is in `$B$`, and every element in `$B$` is in `$C$`. Since every element in `$A$` is in `$B$`, we have to have every element in `$A$` is in `$C$`, therefore `$A\subseteq C$`, as desired.

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For proper subsets, suppose we have two sets such that
$A\subset B$ and $B\subset C$. We already have that
$A\subseteq C$, by the previous argument. Since
$B\subset C$, we have that there must be an element in
$C$ not in $B$. That element would therefore not be
able to be part of $A$, so we must have $A\neq C$. So,
we have $A\subset C$, as desired.
\item[2.2.6] We would have every element in $B$ be in $
\emptyset$. Since $\emptyset$ has no elements, we have
$B$ cannot have any elements, so $B=\emptyset$.
\item[2.2.7] First, for the sake of contradiction,
assume that we have two sets $A$ and $B$ such that
$A\subseteq B$ but $\#(A)>\#(B)$. This would mean we
have more elements in $A$ than in $B$, so we have to
have some element in $A$ not in $B$ (by the pigeonhole
principle). This violiates $A\subseteq B$, so this is a
contradiction. Therefore, we must have $\#(A)\leq \#(B)
$, as desired.

Now, assume we have two sets such that $A\subseteq B$
and $\#(A)=\#(B)$. Then, we have all elements of $A$
are in $B$, and $A$ has the same number of elements as
$B$. This has to mean that $A$ and $B$ have the same
elements, so $A=B$.

Finally, let $A$ and $B$ be two sets such that
$A\subset B$. We have that $\#(A)\leq \#(B)$, as
$A\subset B$ means $A\subseteq B$ as well. However, we
cannot have $\#(A)=\#(B)$ as that would mean $A=B$,
violating $A\subset B$. So, we have $\#(A)<\#(B)$.
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\item[2.2.8] Assume, for the sake of contradiction,
that we have two sets $A$ and $B$ such that $A\subset
B$ and $B\subset A$. This means that $A\subseteq B$ and
$B\subseteq A$, which, by 2.2.4, we have that it means
$A=B$. However, $A\subset B$ (and $B\subset A$) means
that $A\neq B$, so we have a contradiction. So, we
cannot have two sets $A$ and $B$ such that $A\subset B$
and $B\subset A$, as desired.
\item[2.2.9] We have the only subset of $\emptyset$ is
$\{\}$. So, we have $\mathcal{P}(\emptyset)=\{\{\}\}.$
So, the only subsets of $\mathcal{P}(\emptyset)$ are
the empty set and the set containing the empty set.
Therefore, we have
$$\mathcal{P}(\mathcal{P}(\emptyset))=\{\{\},\{\{\}\}
\},$$
which is our answer.
\item[2.2.10] We have $\#(\mathcal{P}(S))$ is the
number of subsets of $S$. We can think of each subset
as follows: for each element in $S$, we choose whether
or not that element is in our desired subset. This
means that for each element in $S$, we have $2$
possibilities. Multiplying it out, we get $2^n$
subsets, so $\#(\mathcal{P}(S))=2^n$, as desired.
\item[2.4] This problem is split into two parts.
\begin{enumerate}
    \item We get that $A\cup B=B$. To show this, first
we show that every element of $A\cup B$ is in $B$.
Since $A\subseteq B$, we have every element of $A$ is
in $B$. Since $A\cup B$ consists of all elements in
either $A$ or $B$, but every element in $A$ is already
in $B$, we have that every element in $A\cup B$ is in
$B$, so $A\cup B\subseteq B$, as desired.

    Now, we show that every element in $B$ is in $A\cup
B$. This is true by definition, as $A\cup B$ is the set
of all elements either one of $A$ or $B$. So,
$B\subseteq A\cup B$.

    Therefore, we have $A\cup B=B$, as desired.
    \item We get that $A\cap B=A$. To show this, first
show that every element of $A\cap B$ is in $A$. Since
$A\cap B$ is the set of all elements in both $A$ and
$B$, we have that $A\cap B\subseteq A$, by definition.

    Now, we show that every element in $A$ is in $A\cap
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B$. Since $A\subseteq B$, we have every element in $A$
is in $B$. Also, $A\cap B$ is the set of all elements
in both $A$ and $B$, which means that $A\cap B$
contains every element in $A$, as each element in $A$
is in $B$. So, we have $A\subseteq A\cap B$.

    Therefore, we have $A\cap B=A$, as desired.
\end{enumerate}
\item[2.5] To prove that
$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C),$$
we have to show that
$$A\cap(B\cup C)\subseteq (A\cap B)\cup(A\cap C)
\textrm{ and }(A\cap B)\cup(A\cap C)\subseteq
A\cap(B\cup C).$$

To show the first statement, let $x\in A\cap (B\cup C)
$. This means that $x\in A$ and $x\in B\cup C$. $x\in
B\cup C$ implies that either $x\in B$ or $x\in C$ (or
both). If $x\in B$, then $x\in A\cap B$, so $x\in
(A\cap B)\cup(A\cap C)$. Similarly, if $x\in C$, then
$x\in A\cap C$, so $x\in (A\cap B)\cup(A\cap C)$. (Note
that $x$ being in both $B$ and $C$ would fit either
case, so we are okay.) This means that
$$A\cap (B\cup C)\subseteq(A\cap B)\cup(A\cap C),$$
as desired.

Now, we show the second statement. Let $x\in (A\cap B)
\cup(A\cap C)$. This means either $x\in A\cap B$ or
$x\in A\cap C$ (or both). If $x\in A\cap B$, then we
have $x\in A$ and $x\in B$, so $x\in B\cup C$ and
therefore is also in $A\cap(B\cup C)$. Similarly, if
$x\in A\cap C$, then we have $x\in A$ and $x\in C$.
This means $x\in B\cup C$, and therefore $x\in
A\cap(B\cup C)$. (Again, note that $x$ being in both
$A\cap B$ and $A\cap C$ would fit either case, so we
are okay.) We therefore have
$$(A\cap B)\cup(A\cap C)\subseteq A\cap (B\cup C),$$
so we have
$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C),$$
as desired.
\item[2.3.1] We get that $A\cup \emptyset=A$. Note that
we have, by definition, $A\subseteq A\cup\emptyset$.
Also, since $A\cup \emptyset$ consists of all elements
in $A$ or in $\emptyset$. Since there are no elements
in $\emptyset$, we have $A\cup\emptyset$ consists of
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the elements in $A$, so $A\cup\emptyset\subseteq A$.
Therefore, we have $A\cup\emptyset =A$, as desired.

Also, we get that $A\cap \emptyset=\emptyset$. Note
that we have, by definition, that
$A\cap\emptyset\subseteq \emptyset$. Since $\emptyset$
is a subset of any set (by 2.2), we have $
\emptyset\subseteq A\cap \emptyset$. So, we have $A\cap
\emptyset=\emptyset$. So, we are done.
\item[2.3.2] First, we show that $A\cup A=A$. We have,
by definition, that $A\subseteq A\cup A$. Also, since
$A\cup A$ consists only of elements both in $A$ and
$A$, we have $A\cup A$ consists of elements in $A$, as
$A$ and $A$ are the same set. So, we have $A\cup
A\subseteq A$. Therefore, we have $A\cup A=A$, as
desired.

Now, we show that $A\cap A=A$. By defintion, we have
$A\cap A\subseteq A$. Now, since $A\cap A$ is the set
of all elements in both $A$ and $A$, which is the same
as the set of all elements in $A$, we have $A\subseteq
A\cap A,$ so $A\cap A=A$, as desired.

Since $A\cup A=A$ and $A\cap A=A$, we have $A\cup
A=A\cap A$, so we get that
$$A\cup A=A\cap A=A,$$
as desired.
\item[2.3.3] We get that $S\cup \{x\}=S$. First, note
that, by definition, $S\subseteq S\cup \{x\}$. Also, we
have $S\cup \{x\}$ is the set of all elements in $S$ or
in $\{x\}$. Since $x\in S$, we have that $S\cup \{x\}$
is the set of all elements in $S$. Therefore, $S\cup\
{x\}\subseteq S$, so $S\cup\{x\}=S$, as desired.

Also, we get that $S\cap\{x\}=S$. To show this, first,
note that, by definition, $S\cap\{x\}\subseteq S$.
Since $S\cap\{x\}$ is the set of all elements in both
$S$ and $\{x\}$, but $x\in S$, we have $S\cap\{x\}$ is
the set of all elements in $S$. So, we have $S\subseteq
S\cap\{x\}$, which means $S\cap\{x\}=S$, as desired.
So, we are done.
\item[2.3.4] To show this, we show each of the
following statements:
$$A\cup(B\cap C)\subseteq (A\cup B)\cap(A\cup C)
\textrm{ and }(A\cup B)\cap(A\cup C)\subseteq
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A\cup(B\cap C).$$

To show the first one, let $x\in A\cup(B\cap C)$. We
have either $x\in A$ or $x\in B\cap C$ (or both). If
$x\in A$, we have $x\in A\cup B$ and $x\in A\cup C$, so
we have $x\in(A\cup B)\cap(A\cup C)$. If we have $x\in
B\cap C$, we have $x\in B$ and $x\in C$. This means
$x\in A\cup B$ and $x\in A\cup C$, so we have
$x\in(A\cup B)\cap(A\cup C)$. Therefore, we have
$A\cup(B\cap C)\subseteq (A\cup B)\cap(A\cup C)$, as
desired.

Now, we show the second statement. Let $x\in(A\cup B)
\cap (A\cup C)$. This means $x\in A\cup B$ and $x\in
A\cup C$. So, we have the following cases:

\textit{Case 1: $x\in A$ and $x\in A$}\newline
In this case, we have $x\in A$. So, we have $x\in
A\cup(B\cap C)$, as desired.

\textit{Case 2: $x\in A$ and $x\in C$}\newline
In this case, we have $x\in A$. So, we have $x\in
A\cup(B\cap C)$, as desired.

\textit{Case 3: $x\in B$ and $x\in A$}\newline
In this case, we have $x\in A$, so we have $x\in
A\cup(B\cap C)$, as desired.

\textit{Case 4: $x\in B$ and $x\in C$}\newline
In this case, we have $x\in B\cap C$, so we have $x\in
A\cup(B\cap C)$, as desired.

Therefore, we get that $(A\cup B)\cap(A\cup C)\subseteq
A\cup(B\cap C),$ which means
$$(A\cup B)\cap(A\cup C)=A\cup(B\cap C),$$
as desired.
\item[2.3.5] This problem is split into two parts.
\begin{enumerate}
    \item We have that $A\cup B$ is the set of all
elements in either $A$ and $B$. Since this is the same
as $A$, we must have every element in $B$ in $A$, so we
have $B\subseteq A$.
    \item We have that $A\cap B$ is the set of all
elements in both $A$ and $B$. Since this is the same as
$A$, we must have every element of $A$ in $B$, so we
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have $A\subseteq B$.
\end{enumerate}
\item[2.3.6] We have $\mathcal{P}(S\cap T)$ is the set
of all subsets of $S\cap T$. This is the same as all
sets in both $\mathcal{P}(S)$ and $\mathcal{P}(T)$.
(This is true as every subset in both cannot use
elements in only one of $S$ or $T$.) So, we have $
\mathcal{P}(S\cap T)$ is the same as $\mathcal{P}(S)
\cap\mathcal{P}(T)$.

However, we still need to prove this! So, first, let
$x\in \mathcal{P}(S\cap T)$. This means that $x$ is a
subset of $S\cap T$, so $x$ consists of elements in
both $S$ and $T$. So, we have $x\in\mathcal{P}(S)$ and
$x\in\mathcal{P}(T)$, and so $\mathcal{P}(S\cap T)
\subseteq\mathcal{P}(S)\cap\mathcal{P}(T).$

Now, let $x\in\mathcal{P}(S)\cap\mathcal{P}(T)$. This
means $x$ must be a subset of both $S$ and $T$. The
only way for this to be possible is for $x$ to consist
only of elements in $S$ and $T$; that is, for $x$ to
consist only of elements in $S\cap T$. This is the same
as saying that $x$ is a subset of $S\cap T$, so
$x\in\mathcal{P}(S\cap T),$ so we have $\mathcal{P}(S)
\cap\mathcal{P}(T)\subseteq\mathcal{P}(S\cap T).$
Therefore, we have that $\mathcal{P}(S\cap
T)=\mathcal{P}(S)\cap\mathcal{P}(T),$ as desired.

Note that $\mathcal{P}(S\cup T)$ involves subsets using
elements in both $S$ and $T$. Since $\mathcal{P}(S)$
and $\mathcal{P}(T)$ only involve elements in one of
$S$ or $T$, we have $\mathcal{P}(S\cup T)$ contains
elements not in $\mathcal{P}(S)$ and $\mathcal{P}(T)$,
so we cannot describe $\mathcal{P}(S\cup T)$ in a
similar way.
\item[2.6] We construct the following truth table:
$$\begin{tabular}{c|c|c}
    $p$ & $\neg p$ & $p\wedge (\neg p)$\\ \hline
    T & F & F\\
    F & T & F
\end{tabular}$$
This shows that '$p$ and (not $p$)' is always false for
any statement $p$, as desired.
\item[2.7] We construct the following truth table:
$$\begin{tabular}{c|c|c|c|c|c|c}
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    $p$ & $q$ & $p\vee q$ & $\neg p$ & $\neg q$ & $
(\neg p)\wedge(\neg q)$ & $\neg((\neg p)\wedge(\neg q))
$\\ \hline
    T & T & \textbf{T} & F & F & F & \textbf{T}\\
    T & F & \textbf{T} & F & T & F & \textbf{T}\\
    F & T & \textbf{T} & T & F & F & \textbf{T}\\
    F & F & \textbf{F} & T & T & T & \textbf{F}
\end{tabular}$$
We bolded each final value to make things easier for
us. As we can see, this means that either $p\vee q$ and
$\neg((\neg p)\wedge(\neg q))$ are both true or both
false, as desired.
\item[2.4.1] This problem is split into $6$ parts.
\begin{enumerate}
    \item This is a statement.
    \item This is a statement.
    \item This is not a statement, because infinity
could be really cool to one person, but not cool to
someone else; it's an opinion.
    \item This is a statement.
    \item This is not a statement, as this asks someone
something; it's a question.
    \item This is not a statement; it's an expression.
\end{enumerate}
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