Lieb's Concavity Theorem

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This is mostly a rehashing James Lee's excellent lecture notes on the proof of Lieb's concavity theorem, which discusses things beyond the proof.

Theorem 1. *Given any $X \in \mathbb{M}_n(\mathbb{C})$ and $s \in [0,1]$, the map*

$$\Gamma(A,B) = \operatorname{Tr}\left(X^* A^s X B^{1-s}\right)$$

is jointly concave on $A, B \succ 0$

In order to prove this, we require this other theorem:

Theorem 2. *For $t \in [0,1]$ the map*

$$F(A,B) = A^s \otimes B^{1-s} \tag{1}$$

is jointly operator concave on $A, B \succ 0$

For those who've forgotten, we say that a map f is operator concave if given any $\lambda \in [0,1]$ we have

$$f(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) \succ \lambda f(A_1, B_1) + (1-\lambda) f(A_2, B_2)$$

At a first glance, it seems to be quite strange that Theorem 2 would imply Theorem 1.

Proposition 1. *If the map F, as defined in Theorem 2, is jointly operator concave then the map Γ, as defined in Theorem 1 is jointly concave, for $A, B \in \mathcal{S}_n^{++}$ i.e. $A, B \succ 0$*

To prove this proposition, we require the following lemma,

Lemma 1. *The map*

$$(A,B) \to (A:B) \triangleq \left(A^{-1} + B^{-1}\right)^{-1}$$

is jointly operator concave.

This is not that difficult to prove. See Lee's notes for an elementary proof. I also think that this could be proven using elementary arguments from Löewner theory, maybe another post about it? Now, armed with this lemma, let us Proposition 2

Proof of Lemma 2. Note that from the Stieltjes representation,

$$
\begin{aligned}
A^p &= \underbrace{\frac{\sin(p\pi)}{\pi}}_{C_p} \int_0^\infty s^{p-1} A(A+sI)^{-1} ds \\
&= C_p \int_0^\infty s^{p-1} \left(\frac{1}{s} \cdot A : I \right) ds
\end{aligned}
\tag{2}
$$

□