

# The Einstein Field Equations (EFE)

Charlie Liu • 11 Nov 2025

The Einstein Field Equations are the foundation of General Relativity, describing how matter and energy curve spacetime, and how that curved spacetime guides the motion of matter and energy.

## The Fundamental Equation

The complete Einstein Field Equation is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

This tensor equation represents a set of 10 coupled, non-linear partial differential equations.

## Left-Hand Side: Spacetime Geometry

This side describes the curvature and geometry of the universe:

- $R_{\mu\nu}$  is the **Ricci Curvature Tensor** - measures how volumes change due to curvature
- $R$  is the **Ricci Scalar** - the trace of the Ricci tensor, representing average curvature
- $g_{\mu\nu}$  is the **Metric Tensor** - defines distances and causality in spacetime
- $\Lambda$  is the **Cosmological Constant** - represents vacuum energy (dark energy)

The combination  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$  is called the **Einstein Tensor**:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

The Einstein tensor is divergence-free ( $\nabla^\mu G_{\mu\nu} = 0$ ), expressing energy-momentum conservation.

## Right-Hand Side: Matter and Energy

This side describes the sources of curvature:

- $T_{\mu\nu}$  is the **Stress-Energy-Momentum Tensor** - encapsulates all energy and momentum sources
  - $T_{00}$ : Energy density ( $\rho c^2$ )
  - $T_{0i}$ : Momentum density and energy flux
  - $T_{ij}$ : Stress (pressure and shear forces)
- $G$  is **Newton's Gravitational Constant**
- $c$  is the **Speed of Light**
- $\frac{8\pi G}{c^4}$  is an extremely small coupling constant showing spacetime's stiffness

## Physical Interpretation

The EFE can be stated as:

*“The curvature of spacetime (described by the Einstein Tensor and Cosmological Constant) is proportional to the matter and energy content (described by the Stress-Energy Tensor).”*

## Vacuum Field Equations

In empty space where  $T_{\mu\nu} = 0$  and  $\Lambda = 0$ , the equations simplify to:

$$R_{\mu\nu} = 0$$

This vacuum equation describes spacetime around spherical masses (Schwarzschild solution) and black holes.

## Key Mathematical Properties

- **Non-Linearity:** The gravitational field carries energy and acts as its own source
- **General Covariance:** Same form in all coordinate systems
- **Conservation Laws:**  $\nabla^\mu T_{\mu\nu} = 0$  ensures local energy-momentum conservation

# Summary of Tensor Components

| Symbol       | Name                   | Role                | Physical Meaning                         |
|--------------|------------------------|---------------------|------------------------------------------|
| $G_{\mu\nu}$ | Einstein Tensor        | Geometry            | Curvature sourced by mass-energy         |
| $R_{\mu\nu}$ | Ricci Curvature Tensor | Geometry            | Volume changes due to curvature          |
| $R$          | Ricci Scalar           | Geometry            | Average curvature                        |
| $g_{\mu\nu}$ | Metric Tensor          | Geometry            | Fundamental potential; defines distances |
| $\Lambda$    | Cosmological Constant  | Geometry/<br>Source | Vacuum energy; cosmic acceleration       |
| $T_{\mu\nu}$ | Stress-Energy Tensor   | Source              | Density of energy and momentum           |
| $G, c$       | Constants              | Coupling            | Strength of gravity's response to matter |

## Conclusion

The Einstein Field Equations elegantly capture the fundamental principle: **mass-energy tells spacetime how to curve, and curved spacetime tells mass-energy how to move.**