# Clutching construction for exotic spheres

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This is a quick note for my own benefit to record an idea I came across during my final year group project. We did this project on exotic spheres, which are manifolds that are *homeomorphic* to $S^n$ but are not *diffeomorphic* to $S^n$. The theorem that we considered was the following:

**Exotic Spheres** There exist 7-manifolds that are homeomorphic but not diffeomorphic to the 7-sphere $S^7$.

The part that I want to note here is the construction of the candidate manifolds for these Exotic Spheres. We did it using a technique called the 'clutching construction' and it is a way of construction vector bundles over spheres. Here is the construction in general:

1. Decompose $S^n$ into its upper and lower hemispheres, which are homeomorphic to disks: $S^n = D^n_- \cup D^n_+$ where the boundary is the intersection $D^n_- \cap D^n_+ = S^{n-1}$.

2. Suppose we have a map $\varphi : S^{n-1} \to GL_k(\mathbb{R})$, then we define the quotient space $E_\varphi = (D^n_- \times \mathbb{R}^k) \sqcup (D^n_+ \times \mathbb{R}^k)/\sim$ where we identify $(x, v) \in \partial D^n_- \times \mathbb{R}^k$ with $(x, \varphi(x)v) \in \partial D^n_+ \times \mathbb{R}^k$. Note that if we apply $\varphi(x) \in GL_k(\mathbb{R})$ to $v \in \mathbb{R}^k$ we get another element of $GL_k(\mathbb{R})$.

So the 'gluing' of these bundles is trivial on the boundary of the hemispheres but the map $\varphi$ introduces a twist in the fibres $\mathbb{R}^k$. This gives us a $k-$dimensional vector bundle

$$
\begin{tikzcd}
\mathbb{R}^k \arrow[r, hook] & E_\varphi \arrow[d, "\pi"] \\
 & S^n = D^n_- \cup D^n_+
\end{tikzcd}
$$

where the map $\pi$ is projection onto the first coordinate, $\pi(x, v) = x$. We call the map $\varphi$ a *clutching function* corresponding to the above vector bundle.

Here are a couple of definitions we will need.

**Definition 1** $GL^+_k(\mathbb{R}) = \{A \in \mathrm{Mat}_k(\mathbb{R}) : \det(A) > 0\}$

**Definition 2** $\mathrm{Vect}^n_+(X)$ is the isomorphism classes of oriented $n$-dimensional vector bundles over the space $X$

**Proposition** Two vector bundles, $E_\varphi$ and $E_\psi$ constructed as above are isomorphic if $\varphi$ and $\psi$ are homotopic

**Proof**
Suppose the maps $\varphi : S^{n-1} \to GL_k(\mathbb{R})$ and $\psi : S^{n-1} \to GL_k(\mathbb{R})$ are homotopic via the homotopy $F : S^{n-1} \times I \to GL_k(\mathbb{R})$. We can use this homotopy as a clutching function to construct a vector bundle. Define the quotient

$$E_F = (D^n_- \times I \times \mathbb{R}^k) \sqcup (D^n_+ \times I \times \mathbb{R}^k)/\sim$$

where the equivalence relation now identifies $(x, t, v) \in \partial D^n_- \times I \times \mathbb{R}^k$ with $(x, t, F(x, t)v) \in \partial D^n_- \times I \times \mathbb{R}^k$. The vector bundle

$$\begin{array}{c} E_F \\ \big\downarrow \\ S^n \times I \end{array}$$

then restricts to $E_\varphi$ over $S^n \times \{0\}$ and to $E_\psi$ over $S^n \times \{1\}$. By Proposition 1.7 in Hatcher's 'Vector Bundles and K-Theory' these vectors bundles are then isomorphic. □

This proposition gives us a bijection from the set of homotopy classes of maps $S^{n-1} \to GL^+_k(\mathbb{R})$ which we denote by $[S^{n-1}, GL^+_k(\mathbb{R})]$ to $\mathrm{Vect}^n_+(S^n)$. Denote this bijection by $\Phi : [S^{n-1}, GL^+_k(\mathbb{R})] \to \mathrm{Vect}^n_+(S^n)$. We can simplify this further by considering the special orthogonal group $SO(k)$ which is a subgroup of $GL^+_k(\mathbb{R})$. We also want to construct bundles over $S^4$ so lets consider the case $k = n = 4$. With these simplifications we get the following bijection:

$$\Psi : [S^3, SO(4)] \to \mathrm{Vect}^4_+(S^4)$$

Now $[S^3, SO(4)]$ is the set of homotopy classes of maps from $S^3 \to SO(4)$ and this is exactly $\pi_3(SO(4))$ so in fact we have a bijection:

$$\Psi : \pi_3(SO(4)) \to \mathrm{Vect}^4_+(S^4)$$

Lets calculate $\pi_3(SO(4))$. We need this propsition from Hatcher's 'Algebraic Topology'

**Proposition 3** A covering space projection $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ for all $n \geq 2$

**Proposition 4** $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$

**Proof** By the previous proposition, it suffices to construct a covering space projection $p : S^3 \times S^3 \to SO(4)$, as then we will have $\pi_3(S^3 \times S^3) \cong \pi_3(SO(4))$. Since $\pi_3(S^3) \cong \mathbb{Z}$ we will then have $\pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \pi_3(SO(4))$.

The following is based on this paper.

Define a homomorphism $\eta : S^3 \times S^3 \to SO(4)$ which takes a pair of unit quaternions $(p, q)$, which we identify with $S^3$, to the linear map $\psi_{p,q} : \mathbb{H} \to \mathbb{H}$ defined by $\psi_{p,q}(x) = pxq^{-1}$. Consider the kernel of this map, $\ker\eta = \{(p, q) \in S^3 \times S^3 \cong \mathbb{H} \times \mathbb{H} : px = xq \text{ for all } x \in \mathbb{H}\}$. Setting $x = 1$, we get $(p, q) \in \ker\eta$ if $p = q$. This implies that $p$ must be in the centre of $\mathbb{H}$, which is equal to $\mathbb{R} \in \mathbb{H}$. But since we are considering unit quaternions, this means that the centre is equal to $\{\pm 1\}$. Therefore $\ker\eta = \{(1, 1), (-1, -1)\}$. So $S^3 \times S^3$ is a double cover of $SO(4)$. We can conclude that $\eta$ is a covering map and apply the previous proposition to get that $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$ □

Note that $SO(n)$ also acts on $D^n$ and $S^{n-1}$. So, if instead of taking the clutching functions over $\mathbb{R}$ we restrict to some discs or spheres, then we get a disc or sphere bundle associated to the vector bundle that arises via the clutching construction.

Therefore to construct candidates for exotic spheres, we will consider $S^3$ bundles over $S^4$ with structure group $SO(4)$. We can classify these vector bundles using $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

To construct these bundles, define a clutching function by

$$f_{i,j} : S^3 \to SO(4)$$

where $f_{i,j}$ takes a unit quaternion $q$ (where we identify the unit quaternions with $S^3$) and sends it to the linear map $v \mapsto q^i v q^j$ as in this original paper by Milnor for some $v \in \mathbb{R}^4 \cong \mathbb{H}$, where we use quaternionic multiplication on the right hand side. Then for each $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$, we get a sphere bundle

$$\begin{array}{ccc} S^3 & \hookrightarrow & E_{i,j} \\ & & \downarrow \\ & & S^4 \end{array}$$

where the candidates for the exotic spheres are the total spaces of these bundles, $E_{ij}$.

We then went on to give conditions on $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $E_{ij}$ is homeomorphic but not diffeomorphic to $S^7$. I may go through these steps in further posts but for now I'll spoil the surprise here in two propositions presented without proof.

**Proposition 5** $E_{i,j}$ is homeomorphic to $S^7$ when $i + j = 1$.

**Proposition 6** Let $E_{ij}$ be a real, oriented, closed manifold as constructed above, with $i + j = 1$ so that $E_{ij}$ is a topological 7-sphere. Suppose $E_{ij}$ is diffeomorphic to the 7-sphere. Then $(i - j)^2 \equiv \pm 1 \mod 7$.