Trying to understand sheaves through constructions

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Recently I have been reading some Algebraic Geometry notes by Andreas Gathmann - available here. While I can accept and broadley understand the definition of a (pre-)sheaf, I wanted to get some idea of what a sheaf might look like in the wild. I found this paper which gives a construction of a sheaf on a graph. This is the example which elucidated sheaves for me finally. But also a sentence in the paper caught my attention: "the *only* structure you need to construct a sheaf is a partial order." I thought about this and I wanted to explore this further so I am going to go through constructing a sheaf on *just* a partial order.

(Pre-)Sheaves *quickly*

In this note I am not going to distinguish between pre-sheaves and sheaves. The distinction will not be useful here, just keep in mind that sometimes I may say sheaf and mean presheaf.

Now the definition in full.

Definition (Sheaf)
A *sheaf* $\mathcal{F}$ of rings (modules, groups, etc, etc….) consists of the following data:

- for every open set $U \subset X$ a ring (module, etc..) $\mathcal{F}(U)$
- for every inclusion $U \subset V \subset X$ a ring (module, etc..) homomorphism $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that
  - $\mathcal{F}(\emptyset) = 0$
  - ρ_{UU} is the identity map for all $U \in X$
  - if $U \subset V \subset W$ we have $\rho_{VU} \circ \rho_{WV} = \rho_{WU}$

The elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U. This defines a presheaf and to get to a sheaf we require that it obeys the following *gluing property*:

If $U \in X$ is an open set and $\{U_i\}$ an open cover of U and $f_i \in \mathcal{F}(U_i)$ are sections for all i such that $f_i \restriction_{U_i \cap U_j} = f_j \restriction_{U_i \cap U_j}$ for all i, j, then there is a unique $f \in \mathcal{F}$ such that $f \restriction_{U_i} = f_i$ for all i.

So essentially this is saying that for each component of our object X, whatever it may be, we assign it a new space, and that we require these spaces to behave nicely. To quote Agrios directly - "Think about it like the mathematical object is a plot of land and a sheaf is like a garden on top of it."

Now lets get to building a simple sheaf which should hopefully illuminate this further.

Sheaf on a partial order

So let's assume we have a partial order $X = \{P, \leq\}$ where $P = \{a, b, c\}$ and let the partial order be $a \leq c$ and $b \leq c$. We can define a topology (The Alexandrov topology) on this by defining the open sets to be the *upper sets* for this partial order, i.e. a subset $U \subseteq X$ is open if $x \in U$ and $x \leq y$ then $y \in U$. Let's quickly confirm that this is a topology. For our case, the open sets are

- $\{\emptyset\}$
- $\{c\}$
- $\{a, c\}$
- $\{b, c\}$
- $\{a, b, c\}$

Clearly any intersection or union of these upper (open) sets is also an upper set, and hence open.

This topology can be generated by the *basic open sets* $U_x = \{y \in P | x \leq y\}$. In our case we have

- $U_a = \{a, c\}$
- $U_b = \{b, c\}$
- $U_c = \{c\}$

Note that if $x \leq y$ then $U_y \subset U_x$:

Let $z \in U_y$ then $y \leq z$ and since $x \leq y$ we have $x \leq y \leq z$. So $z \in U_x$ and $U_y \subset U_x$.

Lets now define our sheaf $\mathcal{F}$. We have to define two things, to each element of our space we must assign new a space, and then define restriction maps between these spaces.

So for the spaces we can define the following:

- $\mathcal{F}(a) = \mathbb{Z}$
- $\mathcal{F}(b) = \mathbb{Z}$
- $\mathcal{F}(c) = \mathbb{Z} \times \mathbb{Z}$

We can define the restriction maps between them to just be the following:

$$\rho_{a\to c} : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \qquad \rho_{b\to c} : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \tag{1}$$

with

$$\rho_{a\to c}(n) = (n, 0) \qquad \rho_{b\to c}(m) = (0, m) \tag{2}$$

Let's calculate the sections of this sheaf, recall the sections are the elements of the spaces we have defined, so we will calculate the elements of $\mathcal{F}(U_i)$ for each of the basic open sets.

- $\mathcal{F}(U_c) = \mathcal{F}(c) = \mathbb{Z} \times \mathbb{Z}$
- $\mathcal{F}(U_b) = \mathcal{F}(\{b, c\})$, so sections are pairs $(x, y) \in \mathcal{F}(b) \times \mathcal{F}(c) = \mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})$ such that $\rho_{b\to c}(x) = y$. With the restriction map above this gives $\mathcal{F}(U_b) = \{(x, (x, 0) : x \in \mathbb{Z}\} \cong \mathbb{Z}$. Hence $\mathcal{F}(U_b) = \mathcal{F}(b) = \mathbb{Z}$
- similar to the above calculation, we have $\mathcal{F}(U_a) = \mathcal{F}(a) = \mathbb{Z}$
- Global sections can be calculated directly as above, but as a shortcut, if we notice that the images of the restriction maps $\rho_{a\to c}$ and $\rho_{b\to c}$ are the coordinate axes: Im($\rho_{a\to c}$)$= \{(x, 0) : x \in \mathbb{Z}\}$ and Im($\rho_{b\to c}$)$= \{(0, y) : y \in \mathbb{Z}\}$ then the only point where these intersect is the origin $\{(0, 0)\}$ so the global sections are trivial.

So now we have spaces assigned to each element of our poset and restriction maps between them - hence a sheaf - and we have calculated the space of sections of this sheaf. The Sheaf axioms weren't explicitly checked, but they are fairly easy to confirm if you wanted.