The running maximum of geometric Brownian motion

ProbablyTrue • 7 Dec 2025

Consider a geometric Brownian motion driven by the Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S(0) = S_0 \tag{1}$$

where B_t is a canonical Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The process S_t is often used as the most basic model of stock prices in real time. Here we will motivate the problem by considering S_t as the population of an invasive species that has found its way into a new habitat where resources are plentiful. The lack of scarcity means that the population will not come close to the carrying capacity of its habitat on the time scale we are interested in, hence we can use Eqn. (1) to model its growth instead of a more complicated logistic type model.

The solution to Eqn. (1) is given by

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right). \tag{2}$$

Since the deterministic growth rate of S_t is μ, it is natural to consider the quantity $X_t = e^{-\mu t} S_t / S_0$. X_t is the ratio between S_t and the population that would result from a model of deterministic exponential growth. We may then ask, what is the probability that S_t ever becomes twice, three times, or r times as large as a its deterministically governed counterpart? In other words, we would like to characterise the function

$$\psi(r,t) := P\left(\sup_{0 \le s \le t} X_s > r\right), \tag{3}$$

paying especial attention to the limit $\lim_{t\to\infty} \psi(r,t)$. We can obtain an easy bound on ψ by noting that X_t is a non-negative martingale with $EX_t = EX_0 = 1$ for all t, hence Doob's martingale inequality for X_t reads

$$\begin{aligned}\psi(r,t) = P\left(\sup_{0 \le s \le t} X_s > r\right) &\le \frac{EX_t}{r} \\ &= \frac{1}{r}.\end{aligned} \tag{4}$$

For $r \leq 1$ this bound is trivial, but,as we will see, it is the best constant-in-t bound for $\psi(\cdot, r)$ when $r > 1$.

If we define

$$Y_t \coloneqq \frac{\log X_t}{\sigma} = B_t - \sigma t/2 \tag{5}$$

then ψ is equivalently written as

$$\psi(r,t) = P\left(\sup_{0\leq s\leq t} Y_s > \frac{\log r}{\sigma}\right). \tag{6}$$

Here let us state two very useful results without proof. The first gives us the density of the running maximum of Brownian motion. This is not so difficult to prove, and I hope to make a separate post about it very soon. The second is a simplified version of Girsanov's theorem which will serve our current needs well enough.

Theorem 1. *(Running maximum of Brownian motion) Let $a > 0$. Then the joint density of B_t and $M_t = \sup_{0\leq s\leq t} B_s$ (the running maximum of B_t) is given by*

$$\nu_t(a,x) \coloneqq P(B_t = x, M_t = a) = \frac{2(2a-x)}{\sqrt{2\pi t^3}} \exp\left(\frac{-(2a-x)^2}{2t}\right). \tag{7}$$

To state the next result we introduce the notation $\langle W\rangle_t$ for the quadratic variation of W_t up to time t, and we write $\langle W, Z\rangle_t$ for the covariation of W_t and Z_t

Theorem 2. *(Girsanov) Let W_t be a continous and $\mathcal{F}_t$-adapted stochastic process. Set*

$$\mathcal{E}_W(t) \coloneqq \exp\left(W_t - \langle W\rangle_t/2\right).$$

Then there exists a measure Q on $(\Omega, \mathcal{F})$ with respect to which $\widetilde{W}_t = B_t - \langle W, B\rangle_t$ is a Brownian motion. Moreoever, Q is aboslutely continuous with respect to P with density $\frac{\mathrm{d}Q}{\mathrm{d}P}|_{\mathcal{F}_t} = \mathcal{E}_W(t)$.

Notice that if we set $W_t = \sigma B_t/2$, then

$$\langle W, B\rangle_t = \frac{\sigma}{2}\langle B, B\rangle_t = \frac{\sigma t}{2},$$

so that $Y_t = B_t - \langle W, B\rangle_t$. We will now use these two results to calculate $\psi(r,t)$. First note that since $\frac{dQ}{dP}|_{\mathcal{F}_t} > 0$ a.s. for all t, $P \ll Q$ and hence $\frac{dP}{dQ}|_{\mathcal{F}_t}$ exists and is given by

$$\frac{dP}{dQ}|_{\mathcal{F}_t} = \left(\frac{dQ}{dF}|_{\mathcal{F}_t}\right)^{-1} = \exp\left(-W_t + \langle W\rangle_t/2\right).$$

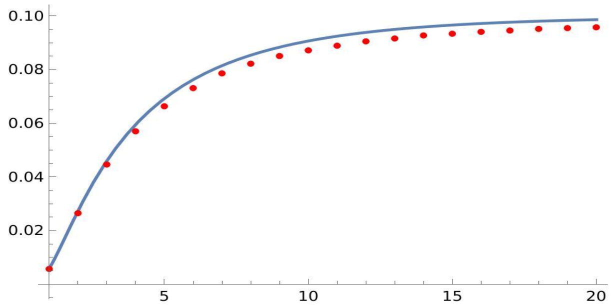
Now we have

$$\psi(r,t) = E^P\left(\mathbb{1}_{\{M_t>r\}}\right) \tag{8}$$

$$= E^Q\left(\mathbb{1}_{\{M_t>r\}}\frac{dP}{dQ}\right) \tag{9}$$

$$= \int_r^\infty \int_{-\infty}^u \nu_t(u,y) \exp\left(-y/2 - \sigma^2 t/8\right)\, \mathrm{d}y\mathrm{d}u \tag{10}$$

Below is a plot of the analytic solution from Eqn. (10) in the case $r = 10, \sigma = 1$ with numerical results obtained by sampling the process Y_t at regular intervals. The numerics systematically underestimate the true probability because sampling the process underestimates the maximum M_t. This raises another question: Does the discrete time process $Y_{k\Delta t},\ k \in \mathbb{N}$ converge to Y_t? In what sense? And how quickly? A topic for another day.



Plot of $\psi(t, 10)$ in blue, with numerics in red dots

It appears from this plot that $\psi(t, 10) \to 1/10$ as $t \to \infty$. I will leave it here for now, but I will post an edit in the next few days to rigorously establish this limit.