The running maximum of geometric Brownian motion

ProbablyTrue • 7 Dec 2025

Consider a geometric Brownian motion driven by the Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S(0) = S_0 \tag{1}$$

where B_t is a canonical Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The process S_t is often used as the most basic model of stock prices in real time. Here we will motivate the problem by considering S_t as the population of an invasive species that has found its way into a new habitat where resources are plentiful. The lack of scarcity means that the population will not come close to the carrying capacity of its habitat on the time scale we are interested in, hence we can use Eqn. (1) to model its growth instead of a more complicated logistic type model.

The stochastic process

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right), \tag{2}$$

is the strong solution to the SDE in Eqn. (1). Since the deterministic growth rate of S_t is μ, it is natural to consider the quantity $X_t = e^{-\mu t} S_t / S_0$. X_t is the ratio between S_t and the population that would result from a model of deterministic exponential growth. We may then ask, what is the probability that S_t ever becomes twice, three times, or r times as large as a its deterministically governed counterpart? In other words, we would like to characterise the function

$$\psi(r,t) \coloneqq P\left(\sup_{0\le s\le t} X_s > r\right), \tag{3}$$

paying especial attention to the limit $\lim_{t\to\infty} \psi(r,t)$. We can obtain an easy bound on ψ by noting that X_t is a non-negative martingale with $EX_t = EX_0 = 1$ for all t, hence Doob's martingale inequality for X_t reads

$$\begin{aligned}\psi(r,t) = P\left(\sup_{0\le s\le t} X_s > r\right) &\le \frac{EX_t}{r} \\ &= \frac{1}{r}.\end{aligned} \tag{4}$$

For $r \le 1$ this bound is trivial, but,as we will see, it is the best constant-in-t bound for $\psi(\cdot, r)$ when $r > 1$. For the time being we will focus on the case $r > 1$.

If we define

$$Y_t := \frac{\log X_t}{\sigma} = B_t - \sigma t/2 \tag{5}$$

then ψ is equivalently written as

$$\psi(r,t) = P\left(\sup_{0 \le s \le t} Y_s > \frac{\log r}{\sigma}\right). \tag{6}$$

Here let us state two very useful results without proof. The first gives us the density of the running maximum of Brownian motion. This is not so difficult to prove, and I hope to make a separate post about it very soon. The second is a simplified version of Girsanov's theorem which will serve our current needs well enough.

Theorem 1. *(Running maximum of Brownian motion) Let $a > 0$. Then the joint density of B_t and $M_t = \sup_{0 \le s \le t} B_s$ (the running maximum of B_t) is given by*

$$\nu_t(a,x) := P(B_t = x, M_t = a) = \frac{2(2a-x)}{\sqrt{2\pi t^3}} \exp\left(\frac{-(2a-x)^2}{2t}\right). \tag{7}$$

To state the next result we introduce the notation $\langle W \rangle_t$ for the quadratic variation of W_t up to time t, and we write $\langle W, Z \rangle_t$ for the covariation of W_t and Z_t

Theorem 2. *(Girsanov) Let W_t be a continous and $\mathcal{F}_t$-adapted stochastic process. Set*

$$\mathcal{E}_W(t) := \exp\left(W_t - \langle W \rangle_t/2\right).$$

Then there exists a measure Q on $(\Omega, \mathcal{F})$ with respect to which $\widetilde{W}_t = B_t - \langle W, B \rangle_t$ is a Brownian motion. Moreoever, Q is aboslutely continuous with respect to P with density $\frac{\mathrm{d}Q}{\mathrm{d}P}|_{\mathcal{F}_t} = \mathcal{E}_W(t)$.

Notice that if we set $W_t = \sigma B_t/2$, then

$$\langle W, B \rangle_t = \frac{\sigma}{2}\langle B, B \rangle_t = \frac{\sigma t}{2},$$

so that $Y_t = B_t - \langle W, B \rangle_t$. We will now use these two results to calculate $\psi(r,t)$. First note that since $\frac{dQ}{dP}|_{\mathcal{F}_t} > 0$ a.s. for all t, $P \ll Q$ and hence, by the Radon-Nikodym theorem, $\frac{dP}{dQ}|_{\mathcal{F}_t}$ exists and is given by

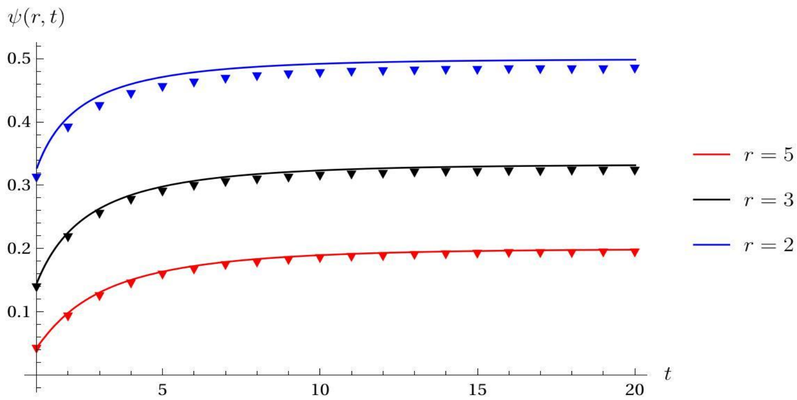
$$\frac{dP}{dQ}|_{\mathcal{F}_t} = \left(\frac{dQ}{dF}|_{\mathcal{F}_t}\right)^{-1} = \exp\left(-W_t + \langle W \rangle_t/2\right).$$

Setting $a = \log(r)/\sigma$ we have

$$\psi(r,t) = E^P\left(\mathbb{1}_{\{M_t > \log(r)/\sigma\}}\right) \tag{8}$$
$$= E^Q\left(\mathbb{1}_{\{M_t > \log(r)/\sigma\}}\frac{dP}{dQ}\right) \tag{9}$$
$$= \int_a^\infty \int_{-\infty}^u \nu_t(u,y)\exp\left(-\sigma y/2 - \sigma^2 t/8\right)\,\mathrm{d}y\mathrm{d}u, \tag{10}$$

where in the last equality we have used the fact that $\nu_t(a,x)$ is the joint density of M_t and Y_t under Q (since Y is a BM under Q).

Below is a plot of the analytic solution from Eqn. (10) in the case $r = 10, \sigma = 1$ with numerical results obtained by sampling the process Y_t at regular intervals. The numerics systematically underestimate the true probability because sampling the process underestimates the maximum M_t. This raises another question: Does the discrete time process $Y_{k\Delta t},\ k \in \mathbb{N}$ converge to Y_t? In what sense? And how quickly? A topic for another day.



Plots of $\psi(r,t)$ with associated numerics marked by triangles of the same colour. We can see that $\psi(r,t)$ is, of course, non-decreasing: If at time t, M_t has had an excursion outside of $[0, \log r]$ then what happens later doesn't matter. If it has not had such an excursion, then it may do in the future. The concavity of $\psi(r,\cdot)$ is intuitively explained when we remember that $B_t \sim \sqrt{t}$ for long times, hence $Y_t \sim -\sigma t/2$ as $t \to \infty$. So if the event $\{\sup_t M_t > a\}$ is observed then M_t probably exceeded a at some moderately small time. Indeed, it appears from the plots that $\psi(r,t) \to 1/r$ as $t \to \infty$. To see that this holds for all $r > 0$ write

$$\psi(r,t) = \int_a^\infty \int_{-\infty}^u \nu_t(u,y)\exp\left(-\sigma y/2 - \sigma^2 t/8\right)\,\mathrm{d}y\mathrm{d}u \tag{11}$$
$$= \int_a^\infty \int_{-\infty}^u \frac{2(2u-y)}{\sqrt{2\pi t^3}} e^{-(2u-y)^2/2t}\exp(-\sigma y/2 - \sigma^2 t/8)\,\mathrm{d}y\mathrm{d}u \tag{12}$$
$$= \int_a^\infty \int_{-\infty}^u \frac{2(2u-y)e^{-\sigma u}}{\sqrt{2\pi t^3}}\exp\{-\frac{(y+\sigma t/2 - 2u)^2}{2t}\}\,\mathrm{d}y\mathrm{d}u. \tag{13}$$

We have skipped a few steps in arriving at the last equality, but these steps just involve completeing the square inside the exponential which is standard. We will now cheat a little by using Mamthematica to evaluate this last integral, which gives

$$
\begin{aligned}
\psi(r,t) &= \int_a^\infty \int_{-\infty}^u \frac{2(2u-y)e^{-\sigma u}}{\sqrt{2\pi t^3}} \exp\{-\frac{(y+\sigma t/2-2u)^2}{2t}\} \,\mathrm{d}y\mathrm{d}u && (14)\\
&= \frac{1}{2}e^{-\sigma a}\left(2-\Phi_c\left(\frac{-2a+\sigma t}{2\sqrt{2t}}\right)\right) - \frac{1}{2}\Phi_c\left(\frac{2a+\sigma t}{2\sqrt{2t}}\right) && (15)\\
&= \frac{1}{2r}\left(2-\Phi_c\left(\frac{-2\log(r)/\sigma+\sigma t}{2\sqrt{2t}}\right)\right) - \frac{1}{2}\Phi_c\left(\frac{2\log(r)/\sigma+\sigma t}{2\sqrt{2t}}\right) && (16)
\end{aligned}
$$

where $\Phi_c(z)$ is the complementary error function

$$\Phi_c(z) := \frac{2}{\sqrt{\pi}}\int_z^\infty e^{u^2/2}\,\mathrm{d}u. \tag{17}$$

Since $\lim_{z\to\infty}\Phi_c(z) = 0$, the limit

$$\lim_{t\to\infty}\psi(r,t) = \frac{1}{r}, \quad r>0 \tag{18}$$

falls out easily from Eqn. (16). We will leave it to the reader to check that ψ is an increasing function of t. The monotonicity of $\psi(r,\cdot)$ and the limit $\lim_{t\to\infty}\psi(r,t) = 1/r$ together imply that the bound in Eqn. (4) is the best bound that is constant in t.

We can also get a handle on the rate of convergence in Eqn. (18) by using the well-known Mill's ratio inequalities,

$$\frac{2}{\sqrt{\pi}}\frac{x}{1+x^2}e^{-x^2/2} < \Phi_c(x) < \frac{2}{x\sqrt{\pi}}e^{-x^2}, \tag{19}$$

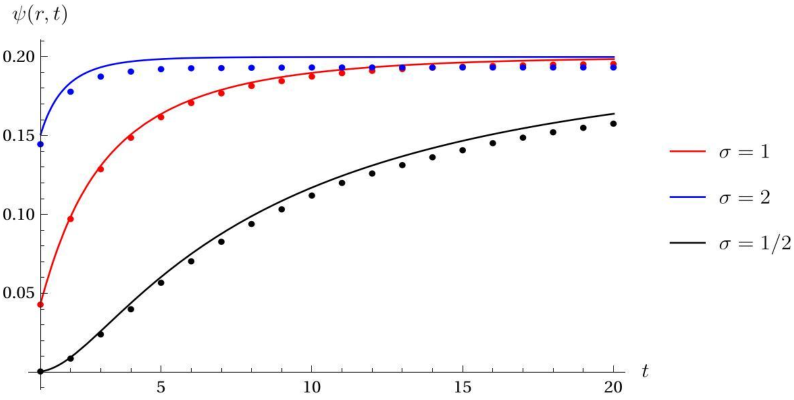
for all $x>0$. Let

$$q_\pm = \frac{\pm 2\log(r)/\sigma+\sigma t}{2\sqrt{2t}} \tag{20}$$

and apply Mill's inequalities to Eqn. (16) to find

$$\frac{1}{2r}\left(2-\frac{2}{q_-\sqrt{\pi}}e^{-q_-^2/2}\right)-\frac{1}{q_+\sqrt{\pi}}e^{-q_+^2/2} < \psi(r,t) < \frac{1}{2r}\left(2-\frac{2}{\pi}\frac{q_-}{1+q_-^2}e^{-q_-^2/2}\right)-\frac{1}{\pi}\frac{}{1 \cdots} \tag{21}$$

Since $q_\pm^2 = \frac{1}{2\sqrt{2}}(\sigma^2 t \pm 8\log r + \frac{\log^2(r)}{\sigma^2 t})$, Eqn. (21) implies that $|\psi(r,t) - 1/r| \sim e^{-\sigma^2 t}/t$. So larger values of σ will result in faster convergence to the limit; see the figure below.



Plots of $\psi(5,t)$ for different values of σ. Solid lines indicate the analytic expression for $\psi(5,t)$, dots of the associated colour indicate numerics. Larger values of σ result in faster convergence.