An Alternative Approach to Limits Using Sequences (Part 1)—Introduction

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Most often, students learn limits rigorously using the epsilon-delta definition. However, quite a few textbooks use or promote an alternative approach (Richmond 1959; Kuratowski 1961; Carpio 1995; Spleight 2016). One such approach involves sequences. For me, this approach is closer to how limits are usually introduced intuitively, though I am not necessarily endorsing this approach—I just want to show how things could be done differently.

First we define what we mean by a sequence.

Definition 1 A sequence is a function whose domain is the set of natural numbers; that is, $f : \mathbb{N} \to A$ for some subset $A \subseteq \mathbb{R}$. The elements of A are called the **terms** of a sequence.

Thus, we do not consider *finite* sequences with only a finite number of terms.

For example, consider the list 1, 1.01, 1.001, 1.0001, … and so on. If we define the function f so that it assigns the nth natural number to the one with $n-1$ 0's in the list ($f(1)=1$, $f(2)=1.01$, … and so on), then f is a sequence. Typically we use the notation a_n instead of $f(n)$ for the terms of the sequence, and $\{a_n\}$ for the sequence itself.

Intuitively, we can "define" the limit of a sequence to be the value that it approaches as n approaches infinity or $n \to \infty$. For some sequences, the limit is obvious. For instance, it is clear that the sequence $\{\frac{1}{2^n}\}$ or $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ approaches 0 as n increases without bound. Similarly, it can be seen using a table of values that as $n \to \infty$,

$$\left\{\frac{4n^2-5n+1}{3n^2+6}\right\} \to \frac{4}{3}.$$

For other sequences though, the limit is not as obvious. Take the sequence

$$1, 0, 1, 0, 1, 0, \ldots$$

Does this sequence eventually approach 1, 0 or 1/2 as a compromise?

A natural way of answering this is through error bounds. To help, we introduce a new definition:

Definition 2 A statement C holds **for sufficiently large** n or **for large** n iff there exists a positive integer N such that for *every* natural number $n > N$, C is true.

Consider the statement $8n^2 + 10n + 5 \leq n^3 - 1$ For small n, like $n = 2$, it is false ($57 > 7$). However, for *every* n larger than 10, the expression on the right "overtakes" the one on the left; thus the inequality holds true. We can therefore pick $N = 10$ so that the statement above is true. Therefore, C *holds for large* n *(*$n > 10$*)*.

Limits of Sequences

Definition 3 A sequence **converges** to L iff there exists a real number L such that for each positive $\epsilon > 0$, there is a corresponding N such that *for large* $n > N$,

$$|a_n - L| < \epsilon$$

holds. We write this as

$$\lim a_n = L.$$

We also say that the **limit** is L.

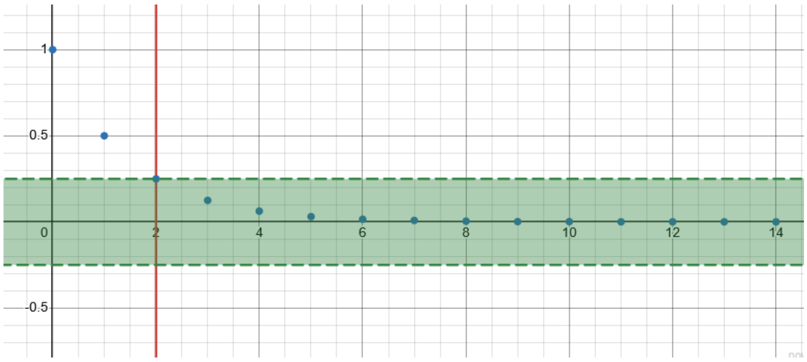
This inequality says that the distance between each a_n and L is no greater than ϵ (Recall that the distance between two real numbers b and a is given by $|b - a|$).

Illustration

We can illustrate this definition by as follows. Take the sequence

$$a_n = \left\{ \frac{1}{2^n} \right\}$$

again, which we intuitively guessed its limit to be 0. Suppose that $\epsilon = 0.25$. That is, the distance between a_n and 0 must be less than 0.25. Below is the graph of the sequence in blue:



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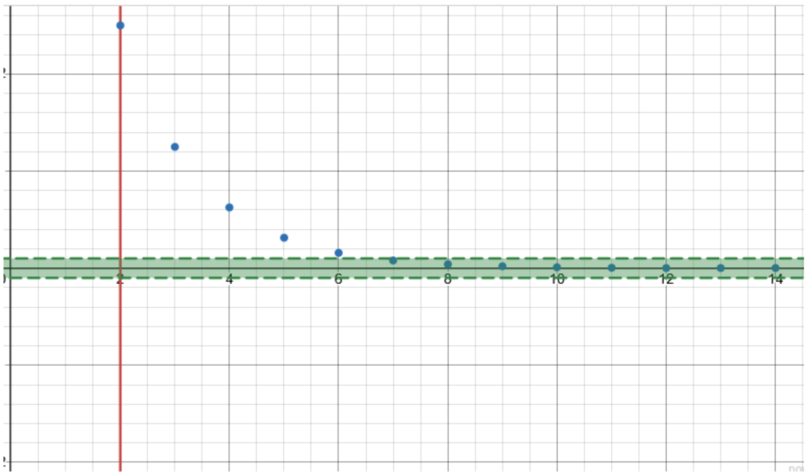
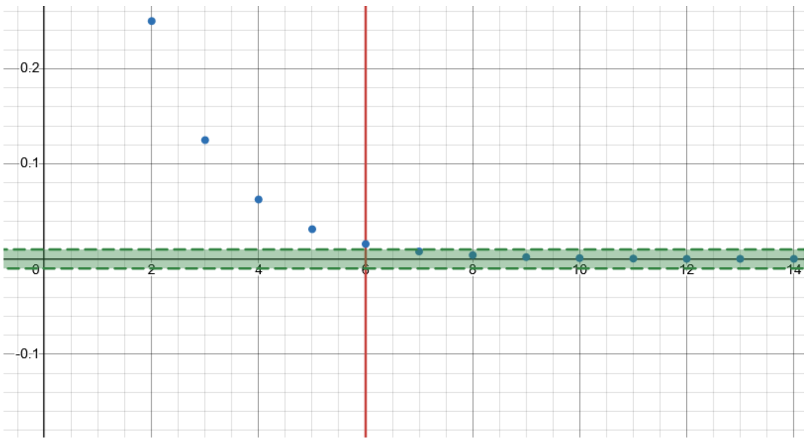
The green area represents all points [not just (n, a_n)] that satisfy our inequality; that is, all points inside the graph of $|y - 0| < 0.25$. Notice that this is so for all sequence points beyond the red line. In other words, all values of a_n satisfy $|a_n - 0| < 0.25$ for all $n > N$, $N = 2$. If we make ϵ smaller, say $\epsilon = 0.01$, we suddenly find that N is too small to make all the blue dots to fit inside the green area:



Notice I changed the y-scaling of the image to make it more visible

Therefore, if the limit is indeed 0, we must find a sufficiently larger N that makes the inequality true. Picking $N = 6$ does the trick:



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Now all the blue dots are in the green area; that is, for all $n > 6$, $|a_n - 0| <$ 0.01\$. For illustration, $|a_7 - 0| = 0.0078125 < 0.01$.

However, this illustration does *not* prove that the limit is actually 0. We show the rigorous proof below:

Proof

Proof Let $\epsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $1 < N\epsilon$, or $\frac{1}{N} < \epsilon$. For all $n \in \mathbb{N}$, if $n > N$, $\frac{1}{n} < \frac{1}{N}$. Furthermore, $n \leq 2^n$ for $n \geq 1$, thus $\frac{1}{n} \leq \frac{1}{2^n}$ (this can be shown by either induction or Bernouili's inequality). Therefore,

$$\left|\frac{1}{2^n} - 0\right| \leq \frac{1}{2^n} < \frac{1}{n} < \frac{1}{N} < \epsilon$$

(Note that we can get rid of the absolute value sign since we're dealing with positive real numbers n anyway). This completes the proof. $\square$

Proof of the Addition Law

The following theorems will be useful in proving limits of sequences, which we will not prove here:

Theorem 4 (Triangle Inequality) For any real numbers a and b,

$$|a + b| \leq |a| + |b|$$

Corollary For any real numbers a and b,

$$|a - b| \geq |a| - |b|$$

We are now able to prove this obvious theorem for limit addition:

Theorem 5 (Addition Law) Let $\{a_n\}$ and $\{b_n\}$ be sequences. If $\lim a_n = L$ and $\lim b_n = M$, then $\lim(a_n + b_n) = L + M$.

Proof Let $\epsilon > 0$. It follows that $\epsilon/2$ is also an arbitrary positive number. Since $\{a_n\}$ and $\{b_n\}$ are convergent, there exist N_1 and N_2 such that for all $n > N_1$

$$|a_n - L| < \epsilon/2$$

and for all $n > N_2$,

$$|b_n - M| < \epsilon/2$$

respectively.

We choose $N = \max\{N_1, N_2\}$ to ensure that both two inequalities above hold. By the triangle inequality,

$$
\begin{aligned}
|(a_n - H_1) + (a_n - H_2)| &= |(a_n + b_n) - (H_1 + H_2)| \\
&\leq |a_n - H_1| + |b_n - H_2| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}
$$

The proof for the multiplication law is more complicated though. I'll leave the proof for the other laws for part 2. $\square$

Limits of General Functions

We can now define the limit of a function in terms of a sequence:

Definition 6 Let f be a real-valued function with real domain D. Let a be a real number with a sequence $\{a_n\}$ converging to it that satisfies $a_n \in D - \{a\}$ for all n (In other words, a is a **limit point** of D).

Then the **limit** of the function f is a real number L iff all sequences $\{x_n\}$ converging to a with each $x_n \in D - \{a\}$ satisfy

$$\lim f(x_n) = L.$$

We denote this as

$$\lim_{x \to a} f(x) = L.$$

We remark a few things.

Remarks

We added the restriction that all sequences must satisfy $a \in D - \{x_n\}$ since we want to study the behavior of x near a, not at a. Supppose we did not. We define a function g such that

$$g(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases} \tag{1}$$

We intuitively guess that $\lim_{x \to 1} g(x)$ should be 0 ($L = 0$). For that to be true, *all* sequences x_n in the domain of g that converge to 1 must satisfy $\lim g(x_n) = 0$. as well. We are then allowed to use the constant sequence $\{1\}$ which converges to 1. But by our modified definition, $\lim g(1) = 3 \neq 0$, which doesn't make sense.

Similarly, we require that a be a limit point of D, since we could also define a function something like $f : [0, 2] \to 1$ and $f(1) = 5$. I could then say that $\lim_{x \to a} f(x) = 6.67 \times 10^{-11}$. which is absurd, but vacuously true.

There's also an interesting approach by Marsden that does away with limits altogether; but that's a topic for another post.

I'll try to post part 2 by next week.

Further reading

Carpio, Harry M. 1995. "An Easy Introduction to Limits." *Matimyás Matematika*, 75–82.

Kuratowski, Kazimierz. 1961. *Introduction to Calculus*. Pergamon Press.

Richmond, Donald E. 1959. *Calculus with Analytic Geometry*. Addison-Wesley Publishing Company.

Spleight, J. M. 2016. *A Sequential Introduction to Real Analysis*. World Scientific Publishing.