The Series Reduction of the integral (lnx)^n

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1 Introduction

When computing integrals of the type $\int (lnx)^n dx$, We generally express them in terms of other integrals using integration by parts which when evaluated several times give the result.

$$\int (lnx)^n dx = x(lnx)^n - n\int (lnx)^{n-1} dx$$

2 Series Reduction

While computing the integral using integration by parts, I recognized a pattern that can be evaluated as a Series.

$$\int (lnx)^n dx = \sum_{k=0}^{n} \frac{n! \cdot x \cdot (lnx)^{n-k} \cdot (-1)^{k+2}}{(n-k)!}$$

$$\int (lnx)^n dx = x \cdot (lnx)^n - n \cdot x \cdot (lnx)^{n-1}, \ldots, (-1)^{n+2} n! \cdot x$$

The sign of the last term which is always an x raised to the power of one, multiplied by the factorial of the power of the integrand depends on the power of the integrand. For even powers, the last term is positive and negative for odd powers.

3 Proof

3.1 For $n = 1$:

To start with the simplest case, when the power of the integrand is one.

$$\int (lnx)^1 dx = \sum_{k=0}^{1} \frac{1! \cdot x \cdot (lnx)^{1-k} \cdot (-1)^{k+2}}{(1-k)!}$$

$$\int (lnx)^1 dx = \frac{1! \cdot x \cdot (lnx)^{1-0} \cdot (-1)^{0+2}}{(1-0)!} + \frac{1! \cdot x \cdot (lnx)^{1-1} \cdot (-1)^{1+2}}{(1-1)!}$$

$$\int (lnx)^1 dx = x \cdot lnx - x + C$$

3.2 Generalized proof:

A more generalized proof requires differentiation of the integral and the series with respect to x.

$$\frac{d}{dx}\int (lnx)^n dx = \frac{d}{dx}\left(\sum_{k=0}^{n} \frac{n! \cdot x \cdot (lnx)^{n-k} \cdot (-1)^{k+2}}{(n-k)!}\right)$$

$$(lnx)^n = \sum_{k=0}^{n} \frac{n! \cdot (-1)^{k+2}}{(n-k)!} \cdot \left(\frac{d}{dx}(x \cdot (lnx)^{n-k})\right)$$

$$(lnx)^n = \sum_{k=0}^{n} \frac{n! \cdot (-1)^{k+2}}{(n-k)!} \cdot \left(\frac{d}{dx}(x) \cdot (lnx)^{n-k} + x \cdot \frac{d}{dx}(lnx)^{n-k}\right)$$

$$(lnx)^n = \sum_{k=0}^{n} \frac{n! \cdot (-1)^{k+2}}{(n-k)!} \cdot \left((lnx)^{n-k} + x \cdot \frac{1}{x} \cdot (n-k) \cdot (lnx)^{n-k-1}\right)$$

$$(lnx)^n = \sum_{k=0}^{n} \frac{n! \cdot (-1)^{k+2}}{(n-k)!} \cdot \big((lnx)^{n-k} + (n-k) \cdot (lnx)^{n-k-1}\big)$$

$$(lnx)^n = \sum_{k=0}^{n} \frac{n! \cdot (-1)^{k+2} \cdot (lnx)^{n-k}}{(n-k)!} + \sum_{k=0}^{n-1} \frac{n! \cdot (-1)^{k+2} \cdot (lnx)^{n-k-1} \cdot (n-k)}{(n-k)!}$$

$$(lnx)^n = \sum_{k=0}^{n} \frac{n! \cdot (-1)^{k+2} \cdot (lnx)^{n-k}}{(n-k)!} + \sum_{k=0}^{n-1} \frac{n! \cdot (-1)^{k+2} \cdot (lnx)^{n-k-1}}{(n-k-1)!}$$

Expanding both series:

$$(lnx)^n = \frac{(-1)^2 \cdot n! \cdot (lnx)^n}{n!} + \frac{(-1)^3 \cdot n! \cdot (lnx)^{n-1}}{(n-1)!} + \frac{(-1)^4 \cdot n! \cdot (lnx)^{n-2}}{(n-2)!}$$

$$, \dots, \frac{(-1)^{n+2} \cdot n! \cdot (lnx)^{n-n}}{(n-n)!} + \left(\frac{(-1)^2 \cdot n! \cdot (lnx)^{n-1}}{(n-1)!} + \frac{(-1)^3 \cdot n! \cdot (lnx)^{n-2}}{(n-2)!}\right.$$

$$\left., \dots, \frac{n! \cdot (-1)^{n-1+2} \cdot (lnx)^{n-(n-1)-1}}{(n-(n-1)-1)!}\right)$$

$$(lnx)^n = \frac{n! \cdot (lnx)^n}{n!} - \frac{n! \cdot (lnx)^{n-1}}{(n-1)!} + \frac{n! \cdot (lnx)^{n-2}}{(n-2)!}$$

$$, \dots, \frac{(-1)^{n+2} \cdot n! \cdot (lnx)^{n-n}}{(n-n)!} + \left(\frac{n! \cdot (lnx)^{n-1}}{(n-1)!} - \frac{n! \cdot (lnx)^{n-2}}{(n-2)!}\right.$$

$$\left., \dots, \frac{n! \cdot (-1)^{n-1+2} \cdot (lnx)^{n-(n-1)-1}}{(n-(n-1)-1)!}\right)$$

$$(lnx)^n = (lnx)^n - n \cdot (lnx)^{n-1} + n \cdot (n-1) \cdot (lnx)^{n-2}, \dots, (-1)^{n+2} \cdot n!$$

$$+n \cdot (lnx)^{n-1} - n \cdot (n-1) \cdot (lnx)^{n-2}, \dots, n! \cdot (-1)^{n+1}$$

$$(lnx)^n = (lnx)^n + n! \cdot ((-1)^{n+1} + (-1)^{n+2})$$

$$(lnx)^n = (lnx)^n + n! \cdot ((-1)^{n+1} + (-1)^{n+1} \cdot (-1))$$

$$(lnx)^n = (lnx)^n + n! \cdot (-1)^{n+1}(1 - 1)$$

$$(lnx)^n = (lnx)^n$$

All the terms cancel out except for the first one showing that the the two sides are equal.

4 Conclusion

Since the Series Reduction formula has been shown to be true for $n = 1$ and in general, for all positive integer multiple powers of the integrand. Therefore, It must be true.