Connections and General Structures on Manifolds

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Connections and General Structures

Connections on a Manifold with Vector Bundles

Definition 1. A **(linear, affine) connection** ∇ on a manifold M is a connection on TM i.e. $\forall\ X, Y \in \mathfrak{X}(M)$, we have (an $\mathbb{R}$-bilinear) $\nabla_X Y \in \mathfrak{X}(M)$ satisfying

$$\nabla_{fX} Y = f\nabla_X Y + (Xf)Y,\ \forall\ f \in C^\infty(M). \tag{1}$$

In this case, $\nabla_X Y$ is the **covariant derivative** of Y along X.

$$\begin{cases} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \\ \nabla_X : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),\ \ X \in \mathfrak{X}(M). \end{cases}$$

Remark. There are some basic concepts to be stated for connections/covariant derivatives.

1. Lie derivative is not a covariant derivative since $\nabla_{fX}Y = f\nabla_X Y$ but $L_{fX}Y = fL_X Y - (Yf)X$.

2. If ∇ and ∇' are connections on M, then $Q := \nabla' - \nabla \in \Omega^1(M, \mathrm{End}(TM))$ s a 1-form with values in the bundle $\mathrm{End}(TM)$.[i] For all $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, write $Q(X,Y) = \nabla'_X Y - \nabla_X Y$. Then $Q(fX, Y) = fQ(X,Y)$ and

$$Q(X, fY) = (f\nabla'_X Y + (Xf)Y) - (f\nabla_X Y + (Xf)Y) = fQ(X,Y).$$

3. In local coordinates (x^μ) on M, write[ii]

$$\nabla_{\partial_\mu}\partial_\nu = \Gamma^\lambda_{\mu\nu}(x)\partial_\lambda,$$

   where $\Gamma^\lambda_{\mu\nu}$ are **Christoffel symbols** of ∇.
   If $x = x^\mu\partial_\mu$, $Y = Y^\nu\partial_\nu$, then

$$\nabla_X Y = X^\mu(Y^\nu\nabla_\mu\partial_\nu + (\partial_\mu Y^\nu)\partial_\nu) = X^\mu(\partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu}Y^\nu)\partial_\lambda.$$

   In tensor calculus we write

$$X^\mu Y^\lambda{}_{;\mu}\partial_\lambda,$$

   where

$$Y^\lambda{}_{;\mu} := \partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu}Y^\nu.$$

   Physicists use this notation as an analogy between covariant derivative and ordinary partial derivatives:

$$Y^\lambda{}_{,\mu} = \partial_\mu Y^\lambda.$$

In the previous paragraphs we say that a connection on a manifold is a “connection on its tangent bundle", or the difference of two connections is an “1-form with values in the bundle $\mathrm{End}(TM)$". These are the concepts in vector bundles, a special type of fibre(fiber) bundles.

Definition 2. A (complex/real) **vector bundle** over a manifold M is a manifold E with a projection $\pi : E \twoheadrightarrow M$ s.t.

1. $\forall\, p \in M$, $E_p := \pi^{-1}(p)$, the **fiber** of p, is a complex/real vector space of dimension r and

2. $\forall\, p \in M$, $\exists$ open neighborhood $U \subset M$ containing p s.t. $\exists$ diffeomorphism $\psi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and

3. $\forall\, q \in U$, $\psi_U : \pi^q \longrightarrow \{q\} \times \mathbb{K}^{-1}$ is a linear isomorphism.

Remark. The transition map on vector bundles are given as follows. If $V \subset M$ is another open neighborhood of p with $\psi_V : \pi^{-1}(V) \longrightarrow V \times \mathbb{K}^r$, then $\forall\, q \in U \cap V$, $\exists\, g_{VU} : U \cap V \longrightarrow \mathrm{GL}(r, \mathbb{K})$ s.t.

$$\begin{aligned} \psi_V \circ \psi_U^{-1} : (U \cap V) \times \mathbb{K}^r &\longrightarrow (U \cap V) \times \mathbb{K}^r \\ (q, v) &\longmapsto (q, g_{VU}(q)v) \end{aligned}$$

Definition 3. If (E, π, M) is a vector bundle, a **section** is a map $s : M \longrightarrow E$ s.t. $\pi \circ s = \mathrm{id}_M$. Let $\Gamma(M, E)$ or simply $\Gamma(E)$ denotes the space of sections.

Remark. In particular, consider TM being a real vector bundle over M, its section $\Gamma(TM)$ is the space of vector field $\mathfrak{X}(M)$.

Definition 4. A **connection ∇ on a vector bundle** $\pi : E \longrightarrow M$ is an $\mathbb{R}$-linear operator

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, s) &\longmapsto \nabla_X s \end{aligned}$$

satisfying

$$\begin{cases} \nabla_{fX} s = f \nabla_X s \\ \nabla_X (fs) = f \nabla_X s + (Xf)s \end{cases}$$

for all $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

Fact 1. *If E and F are vector bundles over M, then so are*

$$E^*,\ E \oplus F,\ E \otimes_{\mathbb{K}} F,\ \mathrm{Hom}_{\mathbb{K}}(E, F),\ \mathrm{End}_{\mathbb{K}}(E),\ \bigwedge^k E$$

with

$$(E^*)_p,\ E_p \oplus F_p,\ E_p \otimes_{\mathbb{K}} F_p,\ \mathrm{Hom}_{\mathbb{K}}(E_p, F_p),\ \mathrm{End}_{\mathbb{K}}(E_p),\ \bigwedge^k E_p.$$

In particular, $\Gamma(\bigwedge^k T^*M) = \Omega^k(M)$.

Definition 5. if $\pi : E \longrightarrow M$ is a vector bundle, a k-form with values in E is a section of $(\bigwedge^k T^*M) \otimes_{\mathbb{R}} E$. We write $\Omega^k(M, E) = \Gamma((\bigwedge^k T^*M) \otimes_{\mathbb{R}} E)$.

Remark. Connections ∇^E on E and ∇^F on F naturally induce connections on

$$E^*,\ E \oplus F,\ E \otimes_{\mathbb{K}} F,\ \mathrm{Hom}_{\mathbb{K}}(E, F),\ \mathrm{End}_{\mathbb{K}}(E),\ \bigwedge^k E$$

by Leibniz rule.

Example 1. Let $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$.

- $\alpha \in \Gamma(E^*)$, then

$$\left(\nabla_X^{E^*} \alpha\right)(s) = X(\alpha(s)) - \alpha\left(\nabla_X^E s\right).$$

- $t \in \Gamma(F)$, then

$$\left(\nabla_X^{E \otimes F}\right)(s \otimes t) = \left(\nabla_X^E s\right) \times t + s \otimes \left(\nabla_X^F t\right).$$

- $A \in \Gamma(\mathrm{End}(E))$, then

$$\left(\nabla_X^{\mathrm{End}(E)} A\right)(s) = \nabla_X^E (As) - A\left(\nabla_X^E s\right).$$

Given a connection ∇ on TM, then connection on T^*M (also denoted by ∇) is given by

$$(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y), \tag{2}$$

for all $X \in \mathfrak{X}(M), \alpha \in \Gamma(T^*M) = \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$.

In local coordinates (X^μ), $\nabla_\mu dx^\lambda = -\Gamma^\lambda_{\mu\nu} dx^\nu$. If an 1-form $\alpha = \alpha_\lambda dx^\lambda$ locally and $X = X^\mu \partial_\mu$, then $\nabla_X \alpha = X^\mu(\partial_\mu \alpha_\nu - \Gamma^\lambda_{\mu\nu}\alpha_\lambda)dx^\nu$. Similarly in tensor calculus,

$$\alpha_{\nu;\mu} := \partial_\mu \alpha_\nu - \Gamma^\lambda_{\mu\nu}\alpha_\lambda.$$

Generally, if $B \in \Gamma\left(\bigotimes^k TM \otimes \bigotimes^\ell T^*M\right)$,

$$B = B^{\lambda_1 \cdots \lambda_k}{}_{\nu_1 \cdots \nu_\ell} \partial_{\lambda_1} \otimes \cdots \otimes \partial_{\lambda_k} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_\ell},$$

then

$$B^{\lambda_1 \cdots \lambda_k}{}_{\nu_1 \cdots \nu_\ell ; \mu} = B^{\lambda_1 \cdots \lambda_k}{}_{\nu_1 \cdots \nu_\ell , \mu} + \sum_{i=1}^{k} B^{\lambda_i \cdots \underset{i\text{-th}}{\rho} \cdots \lambda_k}{}_{\nu_1 \cdots \nu_\ell} \Gamma^{\lambda_i}_{\mu\rho} - \sum_{j=1}^{\ell} B^{\lambda_i \cdots \lambda_k}{}_{\nu_1 \cdots \overset{j\text{-th}}{\sigma} \cdots \nu_\ell} \Gamma$$

Definition 6. Let ∇ be a connection on a manifold.

1. The **torsion** T of ∇ is given by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and
2. the **(Riemann) curvature** R of ∇ is

$$R(X,Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 7. The connection ∇ is **torsion free** if $T = 0$; ∇ is **flat** if $R = 0$.

Lemma 1. *We have* $T \in \Omega^2(M, TM)$ *and* $R \in \Omega^2(M, \mathrm{End}(TM))$.

Skech of the proof.. Obviously both T and R are anti-symmetric.

We need to check that $\forall\ f \in C^\infty(M)$,

$$T(fX, Y) = f(X,Y) = T(X, fY)$$

which is clear; and

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = fR(X,Y)Z.$$

□

Remark. with the concept of torsion and curvature, we have

1. In local coordinates (x^μ), if $T(\partial_\mu, \partial_\nu) = T^\lambda_{\mu\nu}\partial_\lambda$, then $T^\lambda_{\mu\nu} = \Gamma^\lambda_{[\mu,\nu]}$.[iii]
   Let $R(\partial_\mu, \partial_\nu)\partial_\lambda = R_{\mu\nu}{}^\rho{}_\lambda \partial_\rho$, then
   $$R_{\mu\nu}{}^\rho{}_\lambda = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\sigma_{\nu\sigma}\Gamma^\rho_{\mu\lambda}$$
   or one can write it more compactly,
   $$R_{\mu\nu}{}^\rho{}_\lambda = \Gamma^\rho_{[\nu\lambda,\mu]} + \Gamma^\rho_{[\mu\sigma}\Gamma^\sigma_{\nu]\lambda}$$
   where $[a \cdots, \cdots d]$ permutes the indices adjacent to the bracket only. (which are a and d in this case)[iv]

2. In the above, the (Riemann) curvature is defined by the connection on TM, i.e. $R = R^{TM}$. Since $\nabla = \nabla^{TM}$ induces ∇^{T^*M} on T^*M and $\nabla^{\text{End}(TM)}$ on $\text{End}(TM)$, etc. We have $\alpha \in \Omega^1(M) = \Gamma(TM)$, $A \in \Gamma(\text{End}(TM))$,
   $$\begin{cases} R^{T^*M}(X,Y)\alpha = \left([\nabla^{T^*M}_X, \nabla^{T^*M}_Y] - \nabla^{T^*M}_{[X,Y]}\right)\alpha \\ R^{\text{End}(TM)}(X,Y)A = [R(X,Y), A] \end{cases}$$
   Furthermore, $R^{T^*M}(X,Y)\alpha = -{}^tR(X,Y)\alpha \in \Omega^2(M, \text{End}(T^*M))$.

Lemma 2. *Let ∇, ∇' be connections on a manifold M. Let $Q(X,Y) = \nabla' XY - \nabla_X Y$ for all $X, Y \in \mathfrak{X}(M)$. Then $Qin\Gamma(M, \text{Hom}(TM \otimes TM, TM))$ and the torsions T, T' of ∇, ∇' satisfy*

$$T'(X,Y) - T(X,Y) = Q(X,Y) - Q(Y,X), \ \forall\ X, Y \in \mathfrak{X}(M).$$

In particular, ∇, ∇' has the same torsion if and only if $Q(X,Y) = Q(Y,X)$ for all $X, Y \in \mathfrak{X}(M)$.

Proof. From previous discussion we've shown
$Q(fX,Y) = Q(X,fY) = fQ(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.
Thus

$$T'(X,Y) - T(X,Y) = (\nabla'_X Y - \nabla'_Y X - [X,Y]) - (\nabla_X Y - \nabla_Y X - [X,Y]) = Q(X$$

□

Definition 8. Let $\gamma : \mathbb{R} \longrightarrow M$ (or from an open interval $I \subset \mathbb{R}$ containing 0) be a smooth curve on a manifold. Let ∇ be a connection on M. The **parallel transport** of $X_0 \in T_{\gamma(0)}M$ along γ is a set $\{X_s\}$ s.t. $X_s \in T_{\gamma(s)}M$ and $\gamma_{\dot\gamma(s)} X_s = 0$ for all $s \in \mathbb{R}$.
The curve γ is a **geodesic** on M if $\nabla_{\dot\gamma(s)}\dot\gamma(s) = 0$.

Remark. Here we give more notion on geodesics.

1. Note that $\dot\gamma(s)$ and X_s are on the the curve $\gamma(\mathbb{R})$, but $\nabla_{\dot\gamma(s)}X_s$ is well-defined by extending them to a neighborhood of the curve.

2. In local coordinates (x^μ), γ is described by $\gamma^\mu(s)$. Then $X_s = X^\mu(s)\partial_\mu$ is a parallel transport of X_0 if $X^\mu(0) = X_0^\mu$ and $\frac{\mathrm{d}}{\mathrm{d}s}X^\lambda(s) + \Gamma^\lambda_{\mu\nu}(\gamma(s))\frac{\mathrm{d}\gamma^\mu(s)}{\mathrm{d}s}X^\nu(s) = 0$. In particular, γ is a geodesic if

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\gamma^\lambda(s) + \Gamma^\lambda_{\mu\nu}(\gamma(s))\frac{\mathrm{d}\gamma^\mu(s)}{\mathrm{d}s}\frac{\mathrm{d}\gamma^\nu(s)}{\mathrm{d}s} = 0. \tag{3}$$

   For small s, solution $\gamma^\mu(s)$ exists and is unique with the initial conditions $\gamma(0) \in M$, $\dot\gamma(0) \in T_{\gamma(0)}M$.

3. Another connection ∇' on M defines the same geodesic if and only if $Q(X, Y) = (\nabla'_X - \nabla_X)Y$ is anti-symmetric, $Q(X, Y) = -Q(Y, X)$, for all $X, Y \in \mathfrak{X}(M)$.

Corollary 1. *Among the connections on M that define the same geodesics, there is a unique one is torsion-free.*

Sketch of the proof. The uniqueness follows from the Lemma and Remark 3. above. If ∇ is any connection on N, then ∇' is given by $\nabla'_X Y = \nabla_X Y - \frac{1}{2}T(X, Y)$ is a torsion free connection with the same geodesics. □

Torsion, Curvature and Bianchi Identity

Let ∇ be a connection on a manifold M. We have the torsion and connection curvature tensor

$$\begin{cases} T \in \Gamma\left(M, \mathrm{Hom}\left(\bigwedge^2 TM, TM\right)\right), \\ R \in \Gamma\left(M, \mathrm{Hom}\left(\bigwedge^2 TM, \mathrm{End}(TM)\right)\right). \end{cases}$$

Now we have the identities

Theorem 2. *Let T and R be the torsion and curvature tensor of a connection ∇ on a manifold M. Then*

1. ***!st Bianchi identity***

$$R(X,Y)Z + c.p. = T(T(X,Y),Z) + (\nabla_Z T)(X,Y) + c.p.. \tag{4}$$

2. ***2nd Bianchi identity***

$$(\nabla_Z R)(X,Y) + c.p. = R(Z, T(X,Y)) + c.p.. \tag{5}$$

where c.p, *denotes the cyclic permutations.*

Corollary 3. *If ∇ is a torsion-free connection on M, then for all $X, Y \in \mathfrak{X}(M)$,*

$$\begin{cases} R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \\ (\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(X,Z) = 0. \end{cases} \tag{6}$$

Proof of the thoerem.

1. We have

$$\begin{aligned} R(X,Y)Z + \text{c.p.} &= ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z + \text{c.p.} \\ &= \nabla_X \nabla_Y Z - \nabla_Y(\nabla_Z X - [Z,X] - T(Z,X)) - \nabla_{[X,Y]}Z + \text{c} \end{aligned}$$

Note that $\nabla_X \nabla_Y Z - \nabla_Y \nabla_Z X + \text{c.p.} = 0$, thus we have (reordering X, Y, Z by cyclic permutations)

2.

$$\begin{aligned} R(X,Y)Z + \text{c.p.} &= \nabla_Z[X,Y] - \nabla_{[X,Y]}Z + \nabla_Z T(X,Y) + \text{c.p.} \\ &= T(Z,[X,Y]) + \cancel{[Z,[X,Y]]} + (\nabla_Z T)(X,Y) + T(\nabla_Z X, Y) \\ &= T(\nabla_X Y - \nabla_Y X - [X,Y], Z) + (\nabla_Z T)(X,Y) + \text{c.p.} \\ &= T(T(X,Y),Z) + (\nabla_Z T)(X,Y) + \text{c.p.} \end{aligned}$$

We have

$$\begin{aligned} (\nabla_Z R)(X,Y)Z + \text{c.p.} &= \nabla_Z(R(X,Y)) - R(\nabla_Z X, Y) - R(X, \nabla_Z Y) + \text{c.p.} \\ &= [\nabla_Z, R(X,Y)] + R(Z, \nabla_X Y - \nabla_Y X) + \text{c.p} \\ &= \cancel{[\nabla_Z, [\nabla_X, \nabla_Y]]} - [\nabla_Z, \nabla_{[X,Y]}] + R(Z, T(X,Y) + [X \\ &= R(Z, T(X,Y)) - \cancel{\nabla_{[Z,[X,Y]]}} + \text{c.p.} \end{aligned}$$

Note that $(\nabla_Z \circ R(X,Y))W \equiv \nabla_Z(R(X,Y)W) - R(X,Y)\nabla_Z W \equiv [\nabla_Z, R(X,Y)]W$ and every canceling is due to Jacobian identity with cyclic permutations.

□

<u>*Remark*</u>. In local coordinates (X^μ), the Corollay is

$$\begin{cases} R_{\mu\nu}{}^{\rho}{}_{\lambda} + R_{\lambda\mu}{}^{\rho}{}_{\nu} + R_{\nu\lambda}{}^{\rho}{}_{\mu} = 0 \\ \nabla_\lambda R_{\mu\nu}{}^{\rho}{}_{\sigma} + \nabla_\mu R_{\nu\lambda}{}^{\rho}{}_{\sigma} + \nabla_\nu R_{\lambda\mu}{}^{\rho}{}_{\sigma} = 0. \end{cases}$$

Definition 9. Let ∇ be a connection on a manifold M with (Riemannian) curvature (tensor) R. The **Ricci curvature (tensor)** of ∇ is given by

$$\operatorname{Ric}(X, Y) := \operatorname{Tr}(R(\bullet, X)Y), \ \forall X, Y \in \mathfrak{X}(M). \tag{7}$$

Note that $\operatorname{Ric} \in \Gamma(M, T^*M \otimes T^*M)$ (or $(TM \otimes TM)^*$).

Proposition 1. *If ∇ is a torsion free connection on a manifold M, then for all $X, Y, Z \in \mathfrak{X}(M)$,*

1. $\operatorname{Ric}(X, Y) - \operatorname{Ric}(Y, X) = -\operatorname{Tr} R(X, Y)$.
2. $(\nabla_X \operatorname{Ric})(Y, Z) - (\nabla_Y \operatorname{Ric})(X, Z) = \operatorname{Tr}(\nabla_\bullet R)(X, Y)Z$.

Proof. By the Bianchi identities, we have

$$\begin{cases} R(\bullet, X)Y - R(\bullet, Y)X + R(X, Y)(\bullet) = 0 \\ \nabla_X R(\bullet, Y)Z - \nabla_Y R(\bullet, X)Z - (\nabla_\bullet R)(X, Y)Z = 0 \end{cases}$$

Then the result holds by taking the trace of both sides. □

Remark. In local coordinates (x^μ), $R(\partial_\mu, \partial_\nu)\partial_\lambda = R_{\mu\nu}{}^\rho{}_\lambda \partial_\rho$ implies $R_{\mu\nu}$ can represent the Ricci tensors since $R_{\mu\nu} := \operatorname{Ric}(\partial_\mu, \partial_\nu) = R_{\lambda\mu}{}^\lambda{}_\nu$. Thus we have the **(contracted) Bianchi identities**:

$$\begin{cases} R_{\mu\nu} - R_{\nu\mu} + R_{\mu\nu}{}^\lambda{}_\lambda = 0, \\ \nabla_\mu R_{\nu\lambda} - \nabla_\nu R_{\mu\lambda} - \nabla_\rho R_{\mu\nu}{}^\rho{}_\lambda = 0. \end{cases} \tag{8}$$