A Ghost Story

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A Ghost Story-Gauge Fixing

The key question of this chapter is to figure out how to quantize a gauge theory. The first instinct is that, can we obtain the correlation function, or the propagator, as in scalar field theory? Unfortunately, the answer is no., at least not so trivial.

Remark. Consider an Abelian gauge theory over Minkowski spacetime $\mathbb{R}^n$ with the Lagrangian density

$$\mathcal{L}(A) = -\frac{1}{2}F \wedge *F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},\ F = dA,$$

for some $A \in \Omega^1(\mathbb{R}^n)$. To obtain the propagator from $\mathcal{L}$, we can consider the equivalent quadratic form

$$\mathcal{L}_0(A) = \frac{1}{2}A^\mu(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu,$$

Note that $\mathcal{L}$ and $\mathcal{L}_0$ differ by a total divergence, or $\int_{\mathbb{R}^4} dx\mathcal{L} = \int_{\mathbb{R}^4} dx\mathcal{L}_0$ in the sense of weak derivative. Thus the Euler-Lagrange equation is simply

$$(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu = 0.$$

However, this operator behaves badly. To obtain the propagator, we look at momentum space by Fourier transform, which in this case, is the replacement

$$P_\mu(\sqrt{-1}\partial_\nu) \mapsto P_\mu(k_\nu) = -\eta_{\mu\nu}k^2 + k_\mu k_\nu,$$

where P_μ is the polynomial such that $P_\mu(\sqrt{-1}\partial_\nu) = (\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)$ and $k^2 = k_\mu k^\mu$. If $P_\mu(\sqrt{-1}\partial_\nu)$ has inverse then it must be of the form, in momentum space representation,

$$a\eta^{\mu\nu} + bk^\mu k^\nu,$$

for some constants a, b. Then we have

$$(-\eta_{\mu\nu}k^2 + k_\mu k_\nu)(a\eta^{\nu\rho} + bk^\nu k^\rho) = \delta_\mu^\rho.$$

This implies

$$-ak^2\delta_\mu^\rho + ak_\mu k^\rho = \delta_\mu^\rho$$

which has no solution for a.

This means the gauge theory itself has way to many degeneracies so that its impossible to get a unique propagator alone from the Lagrangian $\mathcal{L}$. Geometrically, we have infinitely many gauge orbits in the space of fields. Every orbit represents the same classical solution of the theory, one has to choose a specific orbit then the quantization is consistent. That is,

Proposition 1. *For the Lagrangian density*

$$\mathcal{L}_0(A) = -\frac{1}{2}F \wedge *F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \ F = dA,$$

the following modified Lagrangian density

$$\mathcal{L}(A) = \mathcal{L}_0(A) - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

for some constant α has the momentum representation of the inverse of the propagator:

$$D_{\mu\nu}(k)_\alpha = -\frac{1}{k^2}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2}\right).$$

The proof is just direct computation. Here, we utilize the concept of Lagrange multiplier method, the additional term $(\partial_\mu A^\mu)^2/2\alpha$ does not affect classical solution as long as we choose the *Lorentz gauge* $\partial_\mu A^\mu = 0$.

Remark. In literature we often encounter the two cases

$$\begin{cases} \lim\limits_{\alpha\to 1} D_{\mu\nu}(k)_\alpha = -\dfrac{\eta_{\mu\nu}}{k^2 + \sqrt{-1}\epsilon}, & \text{Feynman gauge,} \\ \lim\limits_{\alpha\to 0} D_{\mu\nu}(k)_\alpha = -\dfrac{\eta_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 + \sqrt{-1}\epsilon}, & \text{Landau gauge.} \end{cases}$$

Now, we show that after choosing a special gauge fixing condition, the propagator is well-defined, and the presumably quantum theory. However, does this procedure depends on the choice of gauge fixing?

Faddeev-Popov Construction

Remark. Without specification, we denote a classical gauge theory by $(P, M, G, \mathcal{F}_M, S)$, a principal G-bundle M over spacetime M with classical (gauge invariant) action S defined on the space of fields $\mathcal{F}_M$. Sometimes we denote $\mathcal{F}_M \times \mathrm{ad}(P)$ or $\mathcal{F}_M \times \Omega^1(M, \mathfrak{g})$ to emphasize the space of gauge fields $\mathrm{ad}(P)$ and matter fields $\mathcal{F}_M$. The space of gauge fields $\mathrm{ad}(P)$ is also frequently denoted by $\mathcal{A}$. The group of gauge transformation is denoted by $\mathcal{G}$.

From the path integral perspective, integrating over the space of gauge fields $\mathcal{A} = \mathrm{ad}(P)$ results in "unregularizable" infinite, since the gauge redundancy is truly uncountable infinite: at every point $x \in M$, we multiply a factor $\mathrm{vol}(G)$. In other words, suppose there is a formal Haar measure defined on $\mathcal{G}$, the reasonable path integral is of the form

$$\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}\tilde{\phi} e^{\sqrt{-1}\tilde{S}[\tilde{\phi}]} = \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{A}} \mathcal{D}\phi e^{\sqrt{-1}S[\phi]}$$

here $\mathcal{D}\tilde{\phi}$ is the pushforward measure of $\mathcal{D}\phi$ and $S = \pi^* \tilde{S}$ where $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$.

Remark. In finite-dimensional case, we have the following well-defined mathematical objects: (see 5)

- Space of gauge fields A, which is modeled by a finite-dimensional affine space;
- Group of gauge transformations G, which is modeled by a finite-dimensional Lie group G;
- Lie group action $G \curvearrowright A$ generated by fundamental vector fields $\{v_a \in \mathfrak{X}(A) \mid a = 1, \cdots, \dim \mathrm{Lie}(G) = n\}$;
- $S \in C^\infty(A)^G$, a g-invariant function and $\mu \in \Omega_c^{\dim A}(A)^G$ a G-invariant top form as the integral measure.

Then we have the following identity

$$\int_A \mu e^{\sqrt{-1}S} = \mathrm{vol}(G) \int_{A/G} \tilde{\mu} e^{\sqrt{-1}\tilde{S}}, \tag{1}$$

where $S = \pi^* \tilde{S}$ and the measure $\tilde{\mu}$ is defined such that $\iota_n \iota_{n-1} \cdots \iota_1 \mu = \pi^* \tilde{\mu}$, here $\pi : A \longrightarrow A/G$ and $\iota_i = \iota_{v_i}$.

We first see how physicists utilize such ideas in practice, 8, 4.

Definition 1. Let $F : \mathcal{A} \longrightarrow \mathrm{Lie}(\mathcal{G})$[i] be a smooth map with the following properties:

- 0 is a regular value of F.
- $F^{-1}(0) \subset \mathcal{A}$ intersects every $\mathcal{G}$-orbit transversally, exactly once.[ii]

Then We say the space of gauge fields satisfying the gauge-fixing condition $F = 0$.

Definition 2. Given the gauge-fixing condition $F = 0$, we define the **Faddeev-Popov determinant** by the formal functional integral:

$$\frac{1}{\Delta_F[A]} = \int_{\mathcal{G}} \mathcal{D}g\delta(F(g \cdot A)) \tag{2}$$

where $\mathcal{D}g$ is the formal Haar measure on $\mathcal{G}$, δ here is the formal Dirac distribution on $\mathcal{G}$ and $g \cdot A$ is the group action fo gauge transformation group $\mathcal{G}$ acts on gauge fields $\mathcal{A}$.

Remark. We assume the solution g to the equation $F(g \cdot A) = 0$ is unique for every fixed $A \in \mathcal{A}$.

Lemma 1. *The Faddeev-Popov determinant is invariant under gauge transformations.*

Proof. Since $\mathcal{D}g$ is the (formal) Haar measure, we have, for all $h \in \mathcal{G}$,
$\mathcal{D}g = \mathcal{D}(gh)$, hence

$$\frac{1}{\Delta_F[h \cdot A]} = \int_{\mathcal{G}} \mathcal{D}g\delta(F(gh \cdot A)) = \int_{\mathcal{G}} \mathcal{D}(gh)\delta(F(gh \cdot A)) = \frac{1}{\Delta_F[A]}.$$

□

Our aim is the replace the integration domain by restriction to the hypersuface $F^{-1}(0)$. This is doable by the following thoerem.

Theorem 1. *If the action S is gauge invariant, then the functional integral*

$$\int_{F^{-1}(0)} \mathcal{D}\mu_F e^{\sqrt{-1}S|_{F^{-1}(0)}} = \frac{1}{\mathrm{vol}(\mathcal{G})} Z, \tag{3}$$

where $\mathcal{D}\mu_F$ is the induced measure on $F^{-1}(0)$. The partition function

$$Z = \int_{\mathcal{A}} \mathcal{D}A e^{\sqrt{-1}S[A]}$$

is defined with respect to gauge invariant measure $\mathcal{D}A$.

Proof. By definition, we have $1 = \Delta_F[A] \int_{\mathcal{G}} \mathcal{D}g\delta(F(g \cdot A))$, then by gauge invariance,

$$\begin{aligned}
Z &= \int_{\mathcal{A}} \mathcal{D}A \left(\int_{\mathcal{G}} \mathcal{D}g\delta(F(g \cdot A))\Delta_F[A] \right) f(A) e^{\sqrt{-1}S[A]} \\
&= \int_{\mathcal{G}} \mathcal{D}g \int_{\mathcal{A}} \mathcal{D}A\delta(F(g \cdot A))\Delta_F[A] f(A) e^{\sqrt{-1}S[A]} \\
&= \int_{\mathcal{G}} \mathcal{D}g \int_{\mathcal{A}} \mathcal{D}(g \cdot A)\delta(F(g \cdot A))\Delta_F[g \cdot A] f(g \cdot A) e^{\sqrt{-1}S[g \cdot A]} \\
&= \int_{\mathcal{G}} \mathcal{D}g \int_{\mathcal{A}} \mathcal{D}(A')\delta(F(A'))\Delta_F[A'] f(A') e^{\sqrt{-1}S[A']} \\
&= \mathrm{vol}(\mathcal{G}) \int_{\mathcal{A}} \mathcal{D}(A)\delta(F(A))\Delta_F[A] f(A) e^{\sqrt{-1}S[A]}
\end{aligned}$$

Note that the first integral has no The rest is to prove that

$$\int_{\mathcal{A}} \mathcal{D}A\delta(F(A))\Delta_F[A]e^{\sqrt{-1}S[A]} = \int_{F^{-1}(0)} \mathcal{D}\mu_F e^{\sqrt{-1}S|_{F^{-1}(0)}}.$$

It suffices to prove that $\Delta_F[A]$ is the (formal) Radon-Nikodym derivative $\mathcal{D}A/\mathcal{D}\mu_F$.[1]

Fix a gauge field $A \in \mathcal{A}$, we parametrize the neighborhood of the orbit through A by $\lambda = \lambda^a T_a \in \Gamma(M, \mathfrak{g})$, $\{T_a \mid a = 1, \cdots, \dim \mathfrak{g}\}$ is a basis of $\mathfrak{g}$, and transverse coordinate y on the slice. Locally we write $A(y) \mapsto g_\lambda \cdot A(y)$ with $A(0) = A$ and $g_0 = e \in \mathcal{G}$. Consider the variation of $F(g_\lambda \cdot A)$ with respect to λ, say

$$F(g_\lambda \cdot A) = F(A) + M_A\lambda + O(\|\lambda\|^2),$$

where M_A is the linear map with the components

$$(M_A)_a = \frac{\delta}{\delta\lambda^a} F(g_\lambda \cdot A)\bigg|_{\lambda=0}.$$

Then we have the identity

$$\int_{\mathrm{Lie}(\mathcal{G})} \mathcal{D}\lambda\delta(F(g_\lambda \cdot A)) = \frac{1}{|\det M_A|}$$

and hence

$$\int_{\mathcal{G}} \mathcal{D}g\delta(F(g \cdot A)) = \frac{1}{\det M_A}$$

(up to normalization). We conclude $\Delta_F[A] = \det M_A$. $\square$

Remark.

- The linearization of the Haar measure reads as follows: Since $g_\lambda = \exp(\lambda)$, $\mathcal{D}g = \mathcal{D}(\exp(\lambda)) = \mu(\lambda)\mathcal{D}\lambda$ for some non-vanishing scaling factor $\mu(\lambda)$ by pushforward of $\lambda \mapsto g_\lambda$ and the left-invariance of the Haar measure. Then we have

$$\begin{aligned}\int_{\mathcal{G}} \mathcal{D}g\delta(F(g\cdot A)) &= \int_{\mathrm{Lie}(\mathcal{G})} \mathcal{D}\lambda\mu(\lambda)\delta(F(A) + M_A\lambda + O(\|\lambda\|^2)) \\ &= \left(\int_{\mathrm{Lie}(\mathcal{G})} \mathcal{D}\lambda\mu(\lambda)\delta(F(A) + M_A\lambda)\right) + O(\|\lambda\|^2)\end{aligned}$$

  While $F(g\cdot A) = 0$ has a unique solution for $g \in \mathcal{G}$, say h. The invariance of Haar measure enable us to consider $F(g\cdot A) = 0$ around $g = e$ by translation by h^{-1} under the integral sign. Then for $F(g_\lambda \cdot A) = 0 \iff F(A) + M_A\lambda + O(\|\lambda\|^2) = 0$, $F(A) = 0$ implies we only consider $\lambda = 0$, that is,

$$\int_{\mathcal{G}} \mathcal{D}g\delta(F(g\cdot A)) = \int_{\mathrm{Lie}(\mathcal{G})} \mathcal{D}\lambda\mu(\lambda)\delta(M_A\lambda) = \frac{\mu(0)}{|\det M_A|}.$$

- Let's translate that variation into geometric language. The gauge group $\mathcal{G}$ acts on $\mathcal{A}$. Its infinitesimal action at A is a linear map

$$\rho_A : T_e\mathcal{G} \longrightarrow T_A\mathcal{A},\ \alpha \mapsto D_A\alpha$$

  where D_A is the covariant derivative with respect to A.

  At A, the composition

$$\mathrm{Lie}(\mathcal{G}) = T_e\mathcal{G} \xrightarrow{\rho_A} T_A\mathcal{A} \xrightarrow{dF_A} T_{F(A)}\,\mathrm{Lie}(\mathcal{G}) \cong \mathrm{Lie}(\mathcal{G})$$

  is how the motion along the gauge orbit away from the slice $F^{-1}(0)$. Then the determinant $\det(dF_A \circ \rho_A)$ is the Faddeev-Popov determinant. Formally we write

$$\Delta_F[A] = \det \left.\frac{\delta F[g\cdot A]}{\delta g}\right|_{g=e}. \tag{4}$$

- The argument above is formal: it uses infinite-dimensional determinants and a formal factorization of the Jacobian. In practice these determinants must be regularized (zeta/heat-kernel/lattice regularization, etc.) and signs/phases must be handled carefully; these issues lead to ghost fields when one exponentiates the determinant.

- Global issues such as Gribov copies (more than one intersection of an orbit with $F^{-1}(0)$) mean the simple formula needs modification in those cases; one may then restrict to a fundamental domain or include an appropriate summation/counting factor.

Now we conclude that

Theorem 2. *From the above assumption, we have*

$$\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}\tilde{A} e^{\sqrt{-1}\tilde{S}} = \int_{F^{-1}(0)} \mathcal{D}\mu_F e^{\sqrt{-1}S|_{F^{-1}(0)}} = \int_{\mathcal{A}} \mathcal{D}A\delta(F(A))\Delta_F[A]e^{\sqrt{-1}S[A]} \tag{5}$$

where $\tilde{S}$ and $\mathcal{D}\tilde{A}$ are the induced action the induced functional measure on $\mathcal{A}/\mathcal{G}$ from $\mathcal{D}A$ on $\mathcal{A}$. Note that the equation holds independent of the choice of F.

Now we express the gauge invariant partition function in the form of functional integral on $\mathcal{A}$ and a geometric interpretation on the slice of gauge orbit $F^{-1}(0)$. For computation purpose, we wish to make the integral $\int_{\mathcal{A}} \mathcal{D}A\delta(F(A))\Delta_F[A]e^{\sqrt{-1}S[A]}$ more convenient to evaluate. This means we have to rewrite the Faddeev-Popov determinant and the Dirac delta distribution.

Remark.

1. Recall that in Grassmannian/Berezinan integral, the determinant of a linear map M can be expressed as

$$\det M = \int_{\Pi(V\oplus V^*)} dvdv^* \exp\left(\sqrt{-1}\,\langle v^*, Mv\rangle\right)$$

   where $\langle\cdot,\cdot\rangle$ is a pairing of V and V^*.

2. Recall that in distribution theory, the Dirac delta distribution admits a Fourier transform representation. Let $f : V \longrightarrow V$ be a sufficiently well-defined function on some Euclidean vector space V

$$\delta(f(x)) = \int_{V^*} dke^{\sqrt{-1}\langle k,f(x)\rangle}$$

   with some normalized Lebesgue measure dk on V^*.

We can, in fact immediately, deduce the following conclusion.

Theorem 3. *From the gauge invariant partition function, we have the functional integration equation:*

$$\int_{\mathcal{A}} \mathcal{D}A\delta(F(A))\Delta_F[A]e^{\sqrt{-1}S[A]} = \int_{\mathcal{A}\times\mathfrak{G}^*\times\Pi(\mathfrak{G}\oplus\mathfrak{G}^*)} \mathcal{D}A\mathcal{D}b\mathcal{D}c\mathcal{D}\bar{c}\exp\left(\sqrt{-1}S_{FP}[A, \tag{6}$$

where $\mathfrak{G} = \mathrm{Lie}(\mathcal{G})$ and the action S_{FP} is

$$S_{FP}[A,b,\bar{c},c] = S[A] + \langle b, F(A)\rangle + S_{gh}[\bar{c},c] \tag{7}$$

*and the **ghost action** $S_{gh}[\bar{c},c]$ is*

$$S_{gh}[\bar{c},c] = \langle\bar{c},(dF_A\circ\rho_A)c\rangle \tag{8}$$

Proof. See the rigorous finite-dimensional version proof of the theorem from Theorem 4.1.4, 5. □

Example 1 (Section 4.1.3, 5)**.** Consider the Yang-Mills theory with a compact Lie group G, i.e. (M, g): (pseudo-)Riemannian manifold,

$$(\mathcal{F}, S_{\mathrm{YM}}) = \left(\Omega^1(M, \mathfrak{g}), \frac{1}{2}\int_M \mathrm{tr}_{\mathrm{Ad}}(F_A \wedge *F_A)\right)$$

and $F_A = D_A A \in \Omega^2(M, \mathfrak{g})$ for $A \in \mathcal{F}$ and $*$ is the Hodge star operator with respect to g. We choose the gauge-fixing condition to be $F(A) = d * A$, then the Faddeev-Popov endomorphism $dF_A = d * D_A$ and the gauge-invariant partition function reads

$$Z = \int_{\mathcal{F}\times\mathfrak{G}^*\times\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} \mathcal{D}A\mathcal{D}b\mathcal{D}c\mathcal{D}\bar{c}\exp\left(\sqrt{-1}S_{\mathrm{FP}}\right),$$

where

$$S_{\mathrm{FP}}[A, b, \bar{c}, c] = S_{\mathrm{YM}}[A] + \int \langle b, d * A\rangle_{\mathfrak{g}} + \int \langle \bar{c}, d * D_A c\rangle_{\mathfrak{g}}.$$

Definition 3. We can interpret the quantum gauge field theory lives in the extended field space

$$\mathcal{A} \times \mathfrak{G}^* \times \Pi(\mathfrak{G} \oplus \mathfrak{G}^*), \;\; \mathfrak{G} = \mathrm{Lie}(\mathcal{G}), \tag{9}$$

with the extended fields:

- The **Faddeev-Popov ghost** c, which is a $\mathfrak{g}$-valued scalar field with fermionic statistics.
- The **anti-ghost** $\bar{c}$, which is a $\mathfrak{g}^*$-valued scalar field with fermionic statistics.
- The **auxiliary field** b, which is $\mathfrak{g}^*$-valued scalar field with bosonic statistics.

No Ghosts in QED

In this section, we will attempt to give a formalism on quantum electrodynamics as a simplified model of quantum gauge theory with ghosts. We follow the derivation of Chapter 12, 4.

Lemma 2. *Let F be a gauge-fixing condition in the sense of the previous section. Choose $f \in \mathfrak{G}$ so that $H(A) := F(A) - f$ is another gauge-fixing condition. Then the gauge invariant partition function reads*

$$Z = \int_{\mathcal{A}\times\Pi(\mathfrak{G}\oplus\mathfrak{G}^*)} \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\exp\left(\sqrt{-1}(S[A] + S_{gh}[c, \bar{c}] + S_{gf}[A])\right), \tag{10}$$

where the ***gauge fixing term****s S_{gf} reads*

$$S_{gf}[A] = G(F(A)), \tag{11}$$

where $G : \mathfrak{G} \longrightarrow \mathbb{C}$ is some smooth functional.

Proof. Note that $\Delta_H[A] = \Delta_F[A]$. We have

$$\int_{\mathcal{A}} \mathcal{D}A \Delta_F[A]\delta(F(A)-f) \exp\big(\sqrt{-1}S[A]\big) = \int_{\mathcal{A}} \mathcal{D}A \Delta_F[A]\delta(F(A)) \exp\big(\sqrt{-1}S[$$

because the integral is independent of the choice of the gauge-fixing condition. Then Z is independent of f, we can consider

$$Z = \int_{\mathfrak{G}} \mathcal{D}f G(f) Z = \int_{\mathfrak{G}} \mathcal{D}f G(f) \int_{\mathcal{A}} \mathcal{D}A \Delta_F[A]\delta(F(A) - f) \exp\big(\sqrt{-1}S[A]\big)$$

up to normalization. Thus

$$\begin{aligned} Z &= \int_{\mathfrak{G}} \mathcal{D}f G(f) \int_{\mathcal{A}} \mathcal{D}A \Delta_F[A]\delta(F(A) - f) \exp\big(\sqrt{-1}S[A]\big) \\ &= \int_{\mathcal{A}} \mathcal{D}A \int_{\mathfrak{G}} \mathcal{D}f G(f) \Delta_F[A]\delta(F(A) - f) \exp\big(\sqrt{-1}S[A]\big) \\ &= \int_{\mathcal{A}} \mathcal{D}A \Delta_F[A] \exp\big(\sqrt{-1}S[A]\big) \int_{\mathfrak{G}} \mathcal{D}f G(f)\delta(F(A) - f) \\ &= \int_{\mathcal{A}} \mathcal{D}A \Delta_F[A] \exp\big(\sqrt{-1}S[A]\big) G(F(A)) \end{aligned}$$

Expanding $\Delta_F[A]$ with ghost fields and we have the result. □

Remark. In QED, a common choice of G is $G(F(A)) = -\sqrt{-1}\int_M d^n x(\partial_\mu A^\mu)^2/2\xi$, $\xi \in \mathbb{R}$, and $F(A) = \partial_\mu A^\mu$. This choice comes from dimensional analysis and the following fact: Let $\mathcal{L}_{\text{FP}} = \mathcal{L}_{\text{QED}} + G(F(A))$, then the equations of motion of $\mathcal{L}_{FP}$ imply that if $F(A) = 0$ somewhere than $F(A) \equiv 0$ identically.

Corollary 4. *With the choice $G(F(A)) = -\sqrt{-1}(\partial_\mu A^\mu)^2/2\xi$, $\xi \in \mathbb{R}$, the partition function of QED becomes*

$$Z = \int_{\mathcal{A}} \mathcal{D}A \exp\left(\sqrt{-1}S_{QED} - \frac{\sqrt{-1}}{2\xi}\int_M d^n x(\partial_\mu A^\mu)^2\right) \tag{12}$$

up to normalization constant.

Proof. It suffices to prove that the Faddeev-Popov determinant is independent of the gauge field. Recall the gauge transformation of $\mathrm{U}(1)$-gauge field is $A \mapsto A + df$ for some smooth function f. Then

$$\Delta_F[A] = \det \frac{\delta}{\delta f}\partial_\mu(f \cdot A)^\mu\bigg|_{f=0} = \det \partial_\mu\partial^\mu.$$

□

The results shows that the Faddeev-Popov determinant is a functional determinant of Laplacian operator. This means the ghost action is a free field theory without coupling and interactions with matter fields. We conclude that QED is a no ghost gauge theory.

Finite-dimensional Justification

In this section, we give a finite-dimensinoal justification of what we done in the Faddeev-Popov procedure. To set up, we consider the following.

- A (compact) Lie group action $G \curvearrowright A$ on some (affine) $\mathbb{C}$-vector space A such that
$$\pi : A \longrightarrow A/G$$
is a trivial principal G-bundle, i.e. $A \overset{\text{diff}}{\cong} G \times A/G$.
- The partition function $Z = Z_S$ is defined as in (1), i.e.
$$Z_S = \int_A \mu e^{\sqrt{-1}S} = \mathrm{vol}(G) \int_{A/G} \tilde{\mu} e^{\sqrt{-1}\tilde{S}},$$
and the notations are given as above.
- The gauge-fixing condition is the same except now we consider the level set $F^{-1}(0)$ intersects every G-orbit transversally N-times for some fix $N \geq 1$. The exact value of N depends on the topology of G and X/G (5).

Now the integral on the quotient X/G can be rewritten as
$$\begin{aligned} Z_S &= \frac{\mathrm{vol}(G)}{N} \int_{F^{-1}(0)} (\pi^* \mu) e^{\sqrt{-1}S|_{F^{-1}(0)}} \\ &= \frac{\mathrm{vol}(G)}{N} \int_A dF^1 \wedge \cdots \wedge dF^{\dim \mathfrak{g}} \wedge \pi^* \mu \delta(F) e^{\sqrt{-1}S} \\ &= \frac{\mathrm{vol}(G)}{N} F^* \delta_0 (e^{\sqrt{-1}S} \pi^* \mu) \end{aligned} \tag{13}$$
where $F^*\delta_0 \in \mathcal{D}'(\mathfrak{g})$ is the Dirac delta distribution $\delta_0 \in \mathcal{D}'(\mathfrak{g})$ pull-backed by F. Note that for a k-cycle $C \subset X$ with $k \leq \dim X$, we have the distributional form $\delta_C^{\dim X - k} \in \Omega^k(X)'$ defined by
$$\delta_C^{\dim X - k} : \omega \mapsto \int_C \omega|_C.$$

We also denote this by
$$\int_C \omega|_C = \int_X \delta_C \wedge \omega$$
for computational convenience.

Lemma 3. *Let Δ_F be a function on A such that, note that $F = \sum_a F^a T_a$ with $F^a \in C^\infty(A)$,*

$$dF^1 \wedge \cdots \wedge dF^{\dim \mathfrak{g}} \wedge \pi^* \mu = \Delta_F \cdot \mu. \tag{14}$$

Then we have the identity

$$\Delta_F(x) = \det_{\mathfrak{g}}(dF_x \circ d\Phi_{e,x}) \tag{15}$$

where $\Phi : G \times A \longrightarrow A$ is the group action.

Remark. Note that the map $FP : dF_x \circ d\Phi_{e,x} \in \mathrm{End}(\mathfrak{g})$ and

$$FP : \mathfrak{g} \xrightarrow{d\Phi_{e,x}} T_x A \xrightarrow{dF_x} \mathfrak{g}.$$

In components, we have

$$FP(x)^a_b = \langle dF^a(x), v_b \rangle_{\mathfrak{g}} = v_b(F^a)|_x$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ denotes the pairing (induced by the bi-invariant metric on G or $\mathfrak{g}$) on $\mathfrak{g}$ and $\mathfrak{g}^*$.
It is also clear that

$$\Delta_F(x) = \det_{\mathfrak{g}} \left.\frac{\delta F(g \cdot x)}{\delta g}\right|_{g=e} := \det_{\mathfrak{g}} \left(\left.\frac{\mathrm{d}}{\mathrm{d}t} F(g_t \cdot x)\right|_{t=0} \right)$$

where g_t is a smooth curve on G parametrized by $t \in \mathbb{R}$ with $g_0 = e \in G$.

Proof of the Lemma. Note that the non-degeneracy of FP is equivalent to $F^{-1}(F(x)) \subset A$ intersecting the G-orbit through x transversally. If the intersection is non-transversal, then $dF^1 \wedge \cdots \wedge dF^{\dim \mathfrak{g}} \wedge \pi^* \mu = 0$ and the statement is trivial.

Let $V = \mathrm{im}\, d\Phi_{e,x} = \mathrm{span}\{v_a(x)\} \subset T_x A$ be the tangent space to the G-orbit through x. Consider the annihilator subspace $\mathrm{Ann}(V) \subset T_x^* A$ and its basis, say $\alpha^1, \cdots, \alpha^\ell$, $\ell = \dim A - \dim G = \dim(A/G)$. Then $\{dF^1(x), \cdots, dF^{\dim \mathfrak{g}}(x), \alpha^1, \cdots, \alpha^\ell\}$ is a basis of $T_x^* A$. W.L.O.G. we may assume $\mu = dF^1 \wedge \cdots \wedge dF^{\dim \mathfrak{g}} \wedge \alpha^1 \wedge \cdots \wedge \alpha^\ell$ By orthogonality of v_a's and α^m's we have, recall that $\iota_a = \iota_{v_a}$,

$$\iota_{\dim \mathfrak{g}} \cdots \iota_1 \mu = \left(\sum_{\sigma \in S_{\dim \mathfrak{g}}} (-1)^\sigma \prod_{a=1}^{\dim \mathfrak{g}} \left\langle dF^a, v_{\sigma(a)} \right\rangle_{\mathfrak{g}} \right) \alpha^1 \wedge \cdots \wedge \alpha^\ell = (\det_{\mathfrak{g}} FP) \alpha^1 \wedge \cdot$$

Wedging with dF^a's we conclude the result. $\square$

Now the following equation is meaningful.

Theorem 5 (Theorem 4.1.3, 5)**.** *Following the above notation, we have*

$$Z_S = \frac{\mathrm{vol}(G)}{N}\int_A \mu\delta(\phi)\Delta_F e^{\sqrt{-1}S} = \frac{\mathrm{vol}(G)}{N}\int_{A\times\mathfrak{g}^*\times\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} \mu_x d\lambda dcd\bar{c}e^{\sqrt{-1}S_{FP}[x,\lambda,c,} \tag{16}$$

where

$$S_{FP}[x,\lambda,c,\bar{c}] = S[x] + \langle\lambda, F(x)\rangle_{\mathfrak{g}} + \langle\bar{c}, FP(x)c\rangle_{\mathfrak{g}}\,. \tag{17}$$

<u>*Remark.*</u> Here we state the Fourier transform and the Grassmannian integral representation precisely.

- The Dirac delta distribution

$$\delta(F(x)) = \int_{\mathfrak{g}^*} d\lambda e^{\sqrt{-1}\langle\lambda,F(x)\rangle_{\mathfrak{g}}}.$$

- The Faddeev-Popov determinant

$$\Delta_F(x) = \int_{\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} dcd\bar{c}e^{\sqrt{-1}\langle\bar{c},FP(x)c\rangle_{\mathfrak{g}}},$$

  where $dcd\bar{c} = \prod_{a=1}^{\dim\mathfrak{g}} dc_a d\bar{c}^a$.

Note that we choose a different normalization from the reference or other literature because the overall scaling the the final path integral is irrelevant.

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1. In fact, it should be $\Delta_F[A] = \mathcal{D}A/(\mathcal{D}g\otimes\mathcal{D}\mu_F)$.↩