Notes on Partition Functions on the Riemann surfaces

JellyShark_aq • 16 Oct 2025

Partition Function on the Riemann surfaces

In this note we follow Lecture 1, part 3 of volume 2 of (Deligne et al. 1999) to calculate Euclidean partition function of free scalar fields with periodic boundary conditions.

The field theory datum $(\mathcal{F}_\Sigma, S, \Sigma)$ are the following

- Σ is a compact Riemannian manifold. (The metric is irrelevant here)
- The space of fields $\mathcal{F}_\Sigma$ is a function class (smooth, distributional, Sobolev, etc.) $\mathrm{Map}(\Sigma, S^1)$ defined on Σ.
- The action is the one of free (bosonic) scalar fields, i.e.
  $$S[\phi] = \frac{\beta}{4\pi}\int_\Sigma d\phi \wedge * d\phi. \tag{1}$$
  The normalization constant $\beta/4\pi$ is merely a convention for now.

First, we see that the space of fields admits the decomposition

$$\mathrm{Map}(\Sigma, S^1) = \mathrm{Map}(\Sigma, \mathbb{R}/2\pi\mathbb{Z}) = \bigsqcup_{\chi\in \mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbb{Z})} \mathrm{Map}(\tilde{\Sigma}, \mathbb{R})_\chi / 2\pi\mathbb{Z},$$

where $\tilde{\Sigma}$ is the universal cover of Σ, which is $\mathbb{C}$ in this case, and $\mathrm{Map}(\tilde{\Sigma}, \mathbb{R})_\chi$ consists of functions $\phi_\chi : \tilde{\Sigma} \longrightarrow \mathbb{R}$ that are $\pi_1(\Sigma)$-equivariant. Explicitly, we have the map $\mathrm{Map}(\Sigma, S^1) \ni \phi \longmapsto \phi_\chi \in \mathrm{Map}(\tilde{\Sigma}, \mathbb{R})_\chi$ with

$$\phi_\chi(ax) = \phi_\chi(x) + \chi(a),\ \ a \in \pi_1(\Sigma).$$

We denote the above function classes by Map to indicate that such decomposition is valid for arbitrary classes.

Note that $\mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbb{Z}) \cong H^1(\Sigma, 2\pi\mathbb{Z})$, then by Hodge decomposition, each ϕ_χ can be uniquely decomposed by

$$\phi_\chi = \int_{\gamma(x_0)} \alpha_h + \psi \equiv \phi_h + \psi,$$

where α_h is the harmonic representative of $\alpha \in H^1(\Sigma, 2\pi\mathbb{Z})$ according to χ.[1] $\gamma(x_0)$ is a (oriented) curve starts from a base point in $x_0 \in \Sigma$ and ψ is a single-valued function on Σ.[2]

The free field action now reads

$$S[\phi_\chi] = \frac{\beta}{4\pi}(\|\alpha_h\|_{L^2}^2 + \langle \psi, -\Delta\psi \rangle_{L^2}),$$

where Δ is the Laplacian. This suggest the following formula

$$\begin{aligned}
\int_{\mathcal{F}_\Sigma} \mathcal{D}\phi e^{-S[\phi]} &= \sum_{\chi \in \mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbb{Z})} \int_{\mathrm{Map}(\tilde{\Sigma}, \mathbb{R})_\chi / 2\pi\mathbb{Z}} \mathcal{D}\phi_\chi e^{-S[\phi_\chi]} \\
&= \sum_{\alpha \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-S[\alpha_h]} \int_{\mathrm{Map}(\Sigma, \mathbb{R})} \mathcal{D}\psi e^{-\beta\langle \psi, -\Delta\psi \rangle_{L^2}/4\pi} \\
&= \sum_{\alpha \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-\beta\|\alpha_h\|_{L^2}^2/4\pi} \int_{\mathrm{Map}(\Sigma, \mathbb{R})} \mathcal{D}\psi e^{-\beta\langle \psi, -\Delta\psi \rangle_{L^2}/4\pi}
\end{aligned}$$

Now the remaining problem is to calculate the Gaussian integral

$$\int_{\mathrm{Map}(\Sigma, \mathbb{R})} \mathcal{D}\psi e^{-\beta\langle \psi, -\Delta\psi \rangle_{L^2}/4\pi}. \tag{2}$$

Thus, we employ the **zeta function regularization**:

Zeta function regularization.

Recall that for finite-dimensional Gaussian integral, we have the result

$$\int_{\mathbb{R}^n} e^{-\langle x, Ax \rangle/2} dx = \sqrt{\frac{(2\pi)^n}{\det A}} \sim \sqrt{\frac{1}{\det A}}$$

(up to normalization) for positive definite $A \in \mathrm{GL}(n, \mathbb{R})$. We define an analogy of the $\det$ term.

Definition 1. Let M be a compact Riemannian manifold. Consider the embedding $C^\infty(M) \hookrightarrow L^2(M)$ and suppose a positive operator A defined on $L^2(M)$ has discrete spectrum $\sigma(A)$, we define the **spectral zeta function** by

$$\zeta_A(z) = \sum_{\lambda \in \sigma(A)}{}' \lambda^{-z} \tag{3}$$

with its meromorphic continuation on $\mathbb{C}$. Note that $\sum' f(x)$ means summation over well-defined $f(z)$.[i]
We denote $\det' A$ by the derivative $\exp(-\zeta_A'(0))$ and refer it as the determinant of A.

The motivation of such definition comes from the finite-dimensional analog: If A is a positive-definite operator on a Euclidean space, then $\zeta_A(z) = \sum_{n=1}^N \lambda_n^{-z}$ and $\zeta_A'(z) = -\sum_{n=1}^N \lambda_n^{-z} \ln \lambda_n$. Then
$\exp(-\zeta_A'(0)) = \exp(\sum_n \ln \lambda_n) = \prod_n \lambda_n = \det A$.

By such definition, we can mimic finite-dimensional Gaussian integral and compute

$$\int_{\mathrm{Map}(\Sigma,\mathbb{R})} \mathcal{D}\psi e^{-\beta\langle\psi,-\Delta\psi\rangle_{L^2}/4\pi} = \sqrt{\frac{2\pi\,\mathrm{vol}(\Sigma)}{\det'(-\beta\Delta/2\pi)}}. \tag{4}$$

Remark. The zero mode of the Laplacian actually contributes the factor $\sqrt{2\pi\,\mathrm{vol}(\Sigma)}$. Let $\psi = \sum_{\lambda\in\sigma(-\Delta)} a_\lambda u_\lambda$, where $a_\lambda \in \mathbb{R}$ and $u_\lambda \in L^2(\Sigma, S^1)$, be a spectral decomposition of $L^2(\Sigma, S^1)$. Since $-\Delta$ is semi-positive definite, we can consider $\psi = \sum_{n=0}^{\infty} a_n u_n$, where $a_n = a_{\lambda_n}$ and so on. We identify $\lambda_0 = 0$ as the 0-th eigenvalue and the degeneracy. The functional integral measure is then understood as[ii]

$$\mathcal{D}\psi = \prod_{n=1}^{\infty} d\psi_n,\ \ \psi_n = a_n/\sqrt{2\pi\,\mathrm{vol}(\Sigma)}.$$

In this case, the functional integral becomes

$$\int_{\mathrm{Map}(\Sigma,\mathbb{R})} \mathcal{D}\psi e^{-\beta\langle\psi,-\Delta\psi\rangle_{L^2}/4\pi} = \prod_{n=0}^{\infty}\int_{\mathbb{R}} d\psi_n \exp\left(-\sum_{n=0}^{\infty}\lambda_n|a_n|^2\right).$$

This "equality" comes from the lattice approximation of functional integral, that is, we regard $\int_{\mathrm{Map}(\Sigma)} = \lim_{n\to\infty} V_\Sigma(n)\int_{\mathrm{Map}(\mathbb{Z}^n)}$, where $1/V_\Sigma(n)$ is the normalization count for $\mathrm{vol}(\Sigma)$. In this case, $V_\Sigma(n) = \sqrt{\mathrm{vol}(\Sigma)}^n$. Thus we can compute the integral as follows:

$$\begin{aligned}
\prod_{n=0}^{\infty}\int_{\mathbb{R}} d\psi_n \exp\left(\sum_{n=0}^{\infty}\beta\lambda_n|a_n|^2/4\pi\right) &= \int_{\mathbb{R}} d\psi_0 \prod_{n\geq 1}\int_{\mathbb{R}} d\psi_n \exp\left(-\sum_{n=1}^{\infty}\beta\lambda_n|a_n|^2/4\pi\right)\\
&= \int_{\mathbb{R}} d\psi_0 \prod_{n\geq 1}\sqrt{\frac{1}{\beta\lambda_n/2\pi}}\\
&= \sqrt{2\pi\,\mathrm{vol}_\Sigma}\prod_{n\geq 1}\sqrt{\frac{1}{\beta\lambda_n/2\pi}}\\
&= \sqrt{\frac{2\pi\,\mathrm{vol}(\Sigma)}{\det'(-\beta\Delta/2\pi)}}.
\end{aligned}$$

We conclude that our final expression with the theorem.

Theorem 1. *For a bosonic Sigma model with S^1-valued defined on $(\mathcal{F}_\Sigma, S, \Sigma) = (\mathrm{Map}(\Sigma, S^1), S[\phi], \Sigma)$ with Σ a compact Riemann manifold, the Euclidean partition function is the form*

$$Z_S(\beta,\Sigma) = \sum_{\alpha\in H^1(\Sigma,2\pi\mathbb{Z})} e^{-\beta\|\alpha_h\|^2_{L^2}/4\pi}\sqrt{\frac{2\pi\,\mathrm{vol}(\Sigma)}{\det'(-\beta\Delta/2\pi)}}. \tag{5}$$

where Δ is the Laplacian of the metric on Σ.

Let us specifically look at 2-dimensional cases, i.e. $\dim_{\mathbb{R}} \Sigma = 2$. We regard Σ as a Riemann surface with genus g.

Proposition 1. *Over the Riemann surface Σ_g with genus g, the partition function defined above has the form:*

$$Z_S(\beta, \Sigma_g) = \exp(-\chi(\Sigma_g)(-3\ln 2\pi + 11\ln \beta/4))\vartheta_{Q(g,\beta)}(\tau, \bar{\tau})\sqrt{\frac{\mathrm{vol}(\Sigma)\det \operatorname{Im}\tau}{\det'(-\Delta)}} \tag{6}$$

where τ is the periodic matrix of Σ and $Q(g,\beta)$ is the lattice

$$Q(g,\beta) = Q_\beta^g = \{(\sqrt{\beta}m + n/\sqrt{\beta})/2, (\sqrt{\beta}m - n/\sqrt{\beta})/2 \mid m, n \in \mathbb{Z}\}^g.$$

The ***theta function*** *ϑ_Q on a lattice $Q \subset E_{s_+,s_-}$ is defined as*

$$\vartheta_Q(\tau, \bar{\tau}) = \sum_{(q_+,q_-)\in Q} \exp\big(\pi\sqrt{-1}((q_+, \tau q_+) - (q_-, \bar{\tau} q_+))\big),$$

and $(\cdot,\cdot)$ is a indefinite bilinear form on the vector space $E_{s_+,s_-} = E_{s_+} \oplus E_{s_-}$. Here, $(\cdot,\cdot)$ has signature $s_+ - s_-$ and is positive and negative definite on E_{s_+} and E_{s_-} respectively.

Proof. Let $(a_i, b_j)_{1\le i,j\le g}$ be a symplectic basis of $H_1(\Sigma, \mathbb{Z})$ with the corresponding basis $(\omega^i)_{1\le i\le g}$ of holomorphic 1-forms $H^{1,0}_{\bar\partial}(\Sigma_g)$. We have

$$\int_{a_i} \omega^j = \delta^{ij}, \ \int_{b_i} \omega^j = \tau^{ij},$$

where $\tau = (\tau^{ij})$ is the periodic matrix. Note that the imaginary part $\operatorname{Im}\tau$ is positive definite. The harmonic form α_h is of the form

$$\alpha_h = \frac{\pi}{\sqrt{-1}}(\bar{\tau}\mathbf{m} + \mathbf{n})^t \operatorname{Im}\tau^{-1}\omega + \text{complex conjugate}.$$

Here $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^g$ gives the harmonic forms in $H^1(\Sigma, 2\pi\mathbb{Z})$ with a_i-periods $-2\pi m_i$ and b_j-periods $2\pi n_j$. The L^2-norm becomes

$$\|\alpha_h\|^2_{L^2} = 2\pi^2\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}\operatorname{Im}\tau^{-1} & \operatorname{Im}\tau^{-1}\operatorname{Re}\tau \\ -\operatorname{Re}\tau\,(\operatorname{Im}\tau)^{-1} & \operatorname{Re}\tau\,(\operatorname{Im}\tau)^{-1}\operatorname{Re}\tau + \operatorname{Im}\tau\end{pmatrix}\begin{pmatrix}\mathbf{m} \\ \mathbf{n}\end{pmatrix}.$$

Then by Poisson summation formula, we have

$$\sum_{\alpha\in H^1(\Sigma,2\pi\mathbb{Z})} e^{-\beta\|\alpha_h\|^2_{L^2}/4\pi} = \beta^{g/2}\sqrt{\det \operatorname{Im}\tau}\,\vartheta_{Q(g,\beta)}(\tau, \bar{\tau}).$$

Inserting this back to the expression we have the desired result. □

Remark. In the reference, Lecture 1, part 3 of volume 2 of [**?**], one can compute the Laplacian determinant term explicitly in lower genus by the analysis of zeta functions. For example, if $g = 1$, the periodic matrix τ is just a complex number in upper half plane, $\tau \in \mathfrak{H}$. Then we have $\det'(-\Delta) = \operatorname{Im}\tau^2 |\eta(\tau)|^4$ where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = e^{\pi\sqrt{-1}\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi\sqrt{-1}n\tau}).$$

Thus the Euclidean partition function of bosonic scalar field theory with S^1-valued is

$$Z_S(\beta, \Sigma) = \sqrt{\frac{\operatorname{vol}(\Sigma)}{\operatorname{Im}\tau}} \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{p_+^2} \bar{q}^{p_-^2},$$

where $q = e^{2\pi\sqrt{-1}\tau}$ and $p_\pm = (\sqrt{\beta}m \pm n/\sqrt{\beta})/2$.

Reference

Deligne, P., P. Etingof, D. S. Freed, et al., eds. 1999. *Quantum fields and strings: A course for mathematicians. Vol. 1, 2.*

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1. This means $[d\phi_\chi] = \alpha \in H^1$↩
2. We have $\int_\gamma \alpha_h = \chi(\gamma)$ for any loop γ. ψ by definition have zero monodromy and hence is single-valued.↩