Some problems of equivariant cohomology and geometry and their solutions

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This article collects problems and solutions from the course: Modern Geometry, Spring 2025 by Siye Wu of National Tsing Hua University. The topic of the course is on group actions on manifolds and equivariant cohomology.

Problem X.

The unit 2-sphere S^2 has a symplectic form which is the standard area form, normalised so that its total area is 4π. The circle group S^1 acts on S^2 by standard rotations around the axis through the north and south poles. Now let $M = S^2 \times S^2$ be equipped with a symplectic form ω from the S^2 factors in an obvious way. The diagonal action of S^1 on M is Hamiltonian and the additive constant in the moment map $\mu : M \longrightarrow \mathbb{R}$ is fixed by the requirement that the average of μ on M is 0. Find the push-forward to $\mathbb{R}$ of the Liouville measure $\omega^2/2$ on M by the moment map μ using two methods: first directly and then by the Duistermaat-Heckman formula. [Hint: The Duistermaat-Heckman formula expresses the Fourier transform

$$\int_{\mathbb{R}} e^{\sqrt{-1}\mu\phi}\omega^2/2$$

of the push-forward measure in terms of the fixed-point data. The inverse Fourier transform of the contribution from each fixed point is ill-defined, but $1/\phi^2$ can be changed to either $1/(\phi + \sqrt{-1}0)^2$ or $1/(\phi - \sqrt{-1}0)^2$.]

Solution.

Let $M = S^2 \times S^2$ with each equipped with the standard area form of total area 4π. Denote by ω_i the symplectic form on the i-th copy of S^2, normalized so that

$$\int_{S^2} \omega_i = 4\pi, \quad i = 1, 2.$$

We consider the diagonal S^1-action on M given by simultaneous rotation about the z-axis on each sphere. On each factor $S^2 \subset \mathbb{R}^3$, with coordinates (x_i, y_i, z_i), this circle action has moment map

$$\mu_i\colon S^2 \longrightarrow [-1, 1], \qquad \mu_i(x_i, y_i, z_i) = z_i, \quad i = 1, 2.$$

Hence the diagonal moment map on

$$M = S^2_{(1)} \times S^2_{(2)}$$

is

$$\mu = \mu_1 + \mu_2 = z_1 + z_2 \colon M = S^2 \times S^2 \longrightarrow [-2, 2].$$

We endow M with the product symplectic form

$$\omega = \omega_1 + \omega_2,$$

so that the Liouville volume form is

$$\frac{\omega^2}{2!} = \omega_1 \wedge \omega_2.$$

We wish to compute the pushforward measure

$$\nu = \mu_*(\omega_1 \wedge \omega_2)$$

on $\mathbb{R}$. Equivalently, we seek a density $\rho(t)$ on $\mathbb{R}$ supported in $[-2, 2]$ so that

$$\nu(dt) = \rho(t)\, dt, \qquad \int_M f\left(\mu(x)\right) \omega_1 \wedge \omega_2 = \int_{-2}^{2} f(t)\, \rho(t)\, dt \quad \forall f \in C_c(\mathbb{R}).$$

We start from computing the pushforward measure.

1. Pushforward on a single sphere: On one sphere $S^2 \subset \mathbb{R}^3$ with coordinates (x, y, z), the moment map $\mu : S^2 \to [-1, 1]$, $\mu(x, y, z) = z$, has level sets $\mu^{-1}(t)$ which are circles of latitude at height $z = t$. In standard spherical coordinates (θ, ϕ) with $z = \cos\theta$, the area form is

   $$\omega = \sin\theta\, d\theta \wedge d\phi, \quad \theta \in [0, \pi], \phi \in [0, 2\pi).$$

   Writing $t = \cos\theta$, we have $dt = -\sin\theta\, d\theta$. Hence $\omega = d\phi \wedge dt$. Consequently, for any test function f,

   $$\int_{S^2} f\left(\mu(x)\right) \omega = \int_{-1}^{1} \left(\int_{\mu^{-1}(t)} d\phi \right) f(t)\, dt = \int_{-1}^{1} (2\pi)\, f(t)\, dt.$$

   However, this expression *already integrates out the angle* but has not accounted for the Jacobian of the change of variable. A more direct check is:

   $$\int_{S^2} f(z)\, \omega = \int_0^{2\pi} \int_0^{\pi} f\left(\cos\theta\right) \sin\theta\, d\theta\, d\phi = 2\pi \int_{-1}^{1} f(t)\, dt.$$

   Equivalently, the pushforward measure $\mu_*(\omega)$ on $[-1, 1]$ has *constant* density 2π. In particular, the total mass is $2\pi \cdot (1 - (-1)) = 4\pi$, as desired.

2. Product manifold and diagonal moment map: On

$$M = S^2_{(1)} \times S^2_{(2)},$$

we write coordinates (z_1, ϕ_1) on the first sphere and (z_2, ϕ_2) on the second, so $\omega_1 = d\phi_1 \wedge dz_1, \omega_2 = d\phi_2 \wedge dz_2$, and $\omega_1 \wedge \omega_2 = d\phi_1\, dz_1 \,\wedge\, d\phi_2\, dz_2$. The diagonal moment map is $\mu = z_1 + z_2$. We view μ as a function $\mu\colon (z_1, z_2) \mapsto t$, and integrate out the angle variables ϕ_1, ϕ_2.

Fix a value $t \in \mathbb{R}$. The fiber $\mu^{-1}(t) \subset M$ is

$$\mu^{-1}(t) = \{(z_1, z_2, \phi_1, \phi_2) : z_1 + z_2 = t, z_i \in [-1, 1], \phi_i \in [0, 2\pi)\}.$$

On that fiber, one has $z_2 = t - z_1$. The volume element $\omega_1 \wedge \omega_2\big|_{\mu^{-1}(t)}$ can be written as

$$d\phi_1\, dz_1 \,\wedge\, d\phi_2\, dz_2 = d\phi_1\, d\phi_2 (\, dz_1 \wedge dz_2\,).$$

But along $\mu^{-1}(t)$, $dz_2 = d(t - z_1) = -\, dz_1$. Thus

$$dz_1 \wedge dz_2 = dz_1 \wedge (-dz_1) = 0,$$

A more direct way is to compute:

$$\int_M \delta\left(\mu(z_1, z_2) - t\right)\, \omega_1 \wedge \omega_2 = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2\pi} \delta(z_1 + z_2 - t)\, d\phi_1\, dz_1\, d\phi_2\, dz_2.$$

First integrate over ϕ_1, ϕ_2 (they range over $[0, 2\pi)$ independently), giving a factor $(2\pi)\,(2\pi) = 4\pi^2$. Then one integrates over $\{(z_1, z_2) \in [-1, 1]^2 : z_1 + z_2 = t\}$. Equivalently, perform the z_2-integral via the Dirac delta measure:

$$\int_{-1}^{1}\int_{-1}^{1} \delta(z_1 + z_2 - t)\, dz_1\, dz_2 = \int_{-1}^{1} \left[\mathbf{1}_{\{z_2 = t - z_1 \in [-1,1]\}}\right]\, dz_1.$$

That indicator is nonzero if and only if

$$z_2 = t - z_1 \in [-1, 1], \quad \text{i.e. } z_1 \in [\,t - 1,\, t + 1\,] \cap [-1, 1].$$

Hence

$$\int_{-1}^{1}\int_{-1}^{1} \delta(z_1 + z_2 - t)\, dz_1\, dz_2 = \text{Length}\left(\,[\,\max(-1,\, t - 1), \min(1,\, t + 1)\,]\,\right).$$

As a function of $t \in [-2, 2]$, one checks

$$\min(1,\, t + 1) - \max(-1,\, t - 1) = \begin{cases} t + 2, & -2 \le t \le 0, \\ 2 - t, & 0 \le t \le 2, \\ 0, & |t| > 2. \end{cases}$$

Therefore, putting everything together,

$$\rho(t) = \int_M \delta(\mu - t)\, \omega_1 \wedge \omega_2 = 4\pi^2\, (2 - |t|)\, \mathbf{1}_{[-2,2]}(t).$$

Consequently,

$$\mu_* \left(\omega_1 \wedge \omega_2\right)(dt) = 4\pi^2 \,\left(2 - |t|\right)\, \mathbf{1}_{[-2,2]}(t) dt.$$

One easily checks the total mass is $\displaystyle\int_{-2}^{2} 4\pi^2 (2 - |t|)\, dt = 16\pi^2$, which equals $\int_M \omega_1 \wedge \omega_2 = (4\pi)(4\pi)$.

Now we derive the result again via Duistermaat-Heckman and inverse Fourier transform:

1. Since $\omega_1 \wedge \omega_2$ is the product measure on $S^2_{(1)} \times S^2_{(2)}$, and $\mu = \mu_1 + \mu_2$, we have

$$Z(s) = \int_{S^2 \times S^2} e^{\sqrt{-1}\, s \mu(z_1, z_2)}\, \omega_1 \wedge \omega_2 = \left(\int_{S^2} e^{\sqrt{-1}\, s\mu(z)}\, \omega\right) \times \left(\int_{S^2} e^{\sqrt{-1}\, s\mu(z)}\, \omega\right).$$

By the localization formula we have

$$\int_{S^2} e^{\sqrt{-1}\, s\mu(z)}\, \omega = 2\pi\, \frac{e^{\sqrt{-1}\, s} - e^{-\sqrt{-1}\, s}}{i\, s} = 4\pi\, \frac{\sin(s)}{s}.$$

Therefore

2. $$Z(s) = \left(4\pi\, \frac{\sin s}{s}\right)^2 = 16\pi^2\, \frac{\sin^2 s}{s^2}.$$

We now recover the density $\rho(t)$ by

$$\rho(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(s)\, e^{-\sqrt{-1}\, s\, t}\, ds = \frac{16\pi^2}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} e^{-\sqrt{-1}\, s\, t}\, ds = 8\pi \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}$$

It is a well-known standard Fourier-transform:

$$\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\, e^{-\sqrt{-1}\, s\, t}\, ds = \pi\, \left(1 - \frac{|t|}{2}\right)\, \mathbf{1}_{[-2,2]}(t).$$

Hence

$$\rho(t) = 8\pi \cdot \pi\, \left(1 - \frac{|t|}{2}\right)\, \mathbf{1}_{[-2,2]}(t) = 4\pi^2\, (2 - |t|)\, \mathbf{1}_{[-2,2]}(t).$$

We conclude again:

$$\mu_* \left(\omega_1 \wedge \omega_2\right)(dt) = 4\pi^2\, (2 - |t|)\, \mathbf{1}_{[-2,2]}(t) dt.$$

Problem XI.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. Each $a \in \mathfrak{g}$ determines a function $f_a \colon \mathcal{O} \to \mathbb{R}$ by

$$f_a(\lambda) = \langle \lambda, a\rangle, \quad \text{for } \lambda \in \mathcal{O}.$$

Also, each $a \in \mathfrak{g}$ defines a vector field X_a on $\mathcal{O}$, induced by the infinitesimal coadjoint action of G on $\mathcal{O}$.

1. Show that

$$X_a f_b = -f_{[a,b]} \quad \text{for all } a, b \in \mathfrak{g},$$

and verify directly that

$$[X_a, X_b] f_c = -X_{[a,b]} f_c \quad \text{for all } a, b, c \in \mathfrak{g}.$$

Now define the Kirillov–Kostant 2-forms $\omega^{\pm} \in \Omega^2(\mathcal{O})$ by

2. $$\omega^{\pm}(X_a, X_b) = \pm f_{[a,b]}.$$

   Show that $\omega^{\pm}$ are well-defined, non-degenerate, and closed. Thus $\omega^{\pm}$ are both symplectic forms on $\mathcal{O}$.

3. Show that the G-action on $\mathcal{O}$ is Hamiltonian with respect to both $\omega^{\pm}$, and that the moment maps $\mu^{\pm} \colon \mathcal{O} \to \mathfrak{g}^*$ are given by

   $$\mu^{\pm}(\lambda) = \mp\lambda.$$

   Hint: Show that $\iota_{X_a}\omega^{\pm} = \pm df_a$.

4. Verify directly that

   $$\{\mu_a^{\pm}, \mu_b^{\pm}\}^{\pm} = \mu_{[a,b]}^{\pm},$$

   where $\{\cdot,\cdot\}^{\pm}$ denote the Poisson brackets associated to the symplectic forms $\omega^{\pm}$, respectively.

Solution.

1. Recall that the vector field X_a on $\mathcal{O}$ is $(X_a)_\lambda = \operatorname{ad}_a^* \lambda$ for all $\lambda \in \mathcal{O}$. Then for any $a, b \in \mathfrak{g}$ and $\lambda \in \mathcal{O}$, we have

   $$\begin{aligned}((X_a)_\lambda f_b) &= f_b(\operatorname{ad}_a^* \lambda)\\ &= \langle \operatorname{ad}_a^* \lambda, b\rangle\\ &= -\langle \lambda, \operatorname{ad}_a b\rangle\\ &= -\langle \lambda, [a,b]\rangle = -f_{[a,b]}.\end{aligned}$$

   Note that we identify $f_b = df_b$ since f_b is linear. Then we have, also, f_a's are linear in a's

$$[X_a, X_b] f_c = X_a X_b f_c - X_b X_a f_c = f_{[a,[b,c]]} - f_{[b,[a,c]]} = f_{[a,[b,c]]-[b,[a,c]]} = f_{[[a,b],c]} = \text{-}$$

   $\forall\ a, b, c \in \mathfrak{g}$ by Jacobi identity.[1]

2. For well-definedness and non-degeneracy, since $\omega^{\pm}(X_b, X_a) = \pm f_{[b,a]} = \mp f_{[a,b]} = -\omega^{\pm}(X_a, X_b)$, it suffice to check the first component.

   - (well-defined) For any $\alpha \in \mathbb{R}$ and $a, b, c \in \mathfrak{g}$, we have, note that X_a's are linear in a's,

$$\omega^\pm(X_a+\alpha X_c, X_b) = \omega^\pm(X_{a+\alpha c}, X_b) = \pm f_{[a+\alpha c,b]} = \pm(f_{[a,b]}+f_{[c,b]}) = \omega^\pm(X_a, X$$

(non-degenerate) If $\omega^\pm(X_a, X_b) = 0$ for all $b \in \mathfrak{g}$, then we have

$$f_{[a,b]}(\lambda) = \langle \lambda, [a,b] \rangle = -\langle \mathrm{ad}_a^* \lambda, b \rangle = 0$$

for all $\lambda \in \mathfrak{g}^*$ and $b \in \mathfrak{g}$. This means $\mathrm{ad}_a^* \lambda = 0$ for all $\lambda \in \mathfrak{g}^*$, thus $a = 0$ and hence $X_a = 0$.

To show $\omega^\pm$ is closed, recall that for any 2-form ω, we have

$$(d\omega)(X,Y,Z) = X\big[\omega(Y,Z)\big] + Y\big[\omega(Z,X)\big] + Z\big[\omega(X,Y)\big] - \omega\big([X,Y],Z\big) - \omega\big([Y$$

Thus we compute

$$\begin{aligned}
&(d\omega^\pm)(X_a, X_b, X_c) \\
&= X_a\big[\omega^\pm(X_b, X_c)\big] + X_b\big[\omega^\pm(X_c, X_a)\big] + X_c\big[\omega^\pm(X_a, X_b)\big] \\
&- \omega^\pm\big([X_a, X_b], X_c\big) - \omega^\pm\big([X_b, X_c], X_a\big) - \omega^\pm\big([X_c, X_a], X_b\big) \\
&= \pm\left(X_a f_{[b,c]} + X_b f_{[c,a]} + X_c f_{[a,b]}\right) \pm \underbrace{\left(f_{[[a,b],c]} + f_{[[b,c],a]} + f_{[[c,a],b]}\right)}_{=0 \text{ by Jacobi identity}} \\
&= \mp\left(f_{[a,[bc]]} + f_{[b,[c,a]]} + f_{[c,[a,b]]}\right) = 0.
\end{aligned}$$

The last line vanishes by Jacobi identity. Thus $\omega^\pm$ are symplectic forms on $\mathcal{O}$.

3. To show the action is Hamiltonian, it suffices to check $L_{X_a}\omega^\pm = 0$ and, for the corresponding moment map $\mu^\pm$, $L_{X_a}\langle \mu^\pm, b\rangle = \langle \mathrm{ad}_a^* \mu^\pm, b\rangle$ for all $a, b \in \mathfrak{g}$.

   ◦ We claim that $\iota_{X_a}\omega^\pm = \pm df_a$. Indeed, for all $b \in \mathfrak{g}$, we have

   $$\iota_{X_a}\omega^\pm(X_b) = \pm f_{[a,b]} = \mp f_{[b,a]} = \pm X_b f_a = \pm df_a(X_b).$$

   Then for Lie derivative we have

   $$L_{X_a}\omega^\pm = d\iota_{X_a}\omega^\pm + \iota_{X_a} d\omega^\pm = d\iota_{X_a}\omega^\pm = \pm d^2 f_a = 0.$$

   Thus consider $\langle \mu^\pm, a\rangle := \mp f_a$ for all $a \in \mathfrak{g}$ we obtain the moment map of $\omega^\pm$.

   ◦ The Lie derivative of the moment map μ reads

   $$L_{X_a}\left\langle \mu^\pm, b\right\rangle = \mp\iota_{X_a} df_b = \mp f_{[b,a]}.$$

   Meanwhile, $\langle \mathrm{ad}_a^* \mu^\pm(\lambda), b\rangle = -\langle \mu^\pm(\lambda), [a,b]\rangle = \pm f_{[a,b]}(\lambda)$ for all $\lambda \in \mathcal{O}$. We conclude $L_{X_a}\langle \mu^\pm, b\rangle = \langle \mathrm{ad}_a^* \mu^\pm, b\rangle$.

Now, for each $\lambda \in \mathcal{O}$, we have

$$\left\langle \mu^\pm(\lambda), a\right\rangle = \mp f_a(\lambda) = \langle \mp\lambda, a\rangle, \ \forall\ a \in \mathfrak{g},\ \lambda \in \mathcal{O}.$$

We conclude $\mu^{\pm}(\lambda) = \mp\lambda$ for all $\lambda \in \mathcal{O}$.

4. Recall the Poisson bracket $\{f, g\}^{\pm} = -\omega^{\pm}(H_f, H_g) = H_g f$ where $\iota_{H_f}\omega^{\pm} = -df$, etc. Note that $\mu_a^{\pm} = \langle \mu^{\pm}, a \rangle = \mp df_a$, hence

$$\iota_{H_{\mu_a^{\pm}}}\omega^{\pm} = -d\mu_a^{\pm} = \pm df_a = \iota_{X_a}\omega^{\pm} \implies H_{\mu_a^{\pm}} = X_a.$$

Then the Poisson bracket of the moment map is,

$$\{\mu_a^{\pm}, \mu_b^{\pm}\}^{\pm} = H_{\mu_b^{\pm}}\mu_a^{\pm} = \mp X_b f_a = \pm f_{[b,a]} = \mp f_{[a,b]} = \mu_{[a,b]}^{\pm}.$$

Problem XII.

1. Consider the defining representation of $G = \mathrm{SU}(2)$ on $\mathbb{C}^2$. For any integer $n \geq 0$, let $V(n)$ be the nth symmetric power of $\mathbb{C}^2$, which is also a representation of G. A maximal torus of G is the subgroup $T \subset G$ consisting of the diagonal matrices in G. Find the weight space decomposition of $V(n)$ according to representations of T.

2. It is known that the complexification $G^{\mathbb{C}}$ of G is $\mathrm{SL}(2, \mathbb{C})$. Let B^- be the subgroup of $G^{\mathbb{C}}$ consisting of lower-triangular matrices. Then B^- acts on $G^{\mathbb{C}}$ holomorphically by right multiplication, and $G^{\mathbb{C}}/B^-$ is a complex space. Show that $G^{\mathbb{C}}/B^-$ is holomorphically diffeomorphic to $\mathbb{CP}^1$, and that there is a holomorphic left $G^{\mathbb{C}}$-action on $\mathbb{CP}^1$.

3. For $n \geq 0$, let $\rho_n : B^- \to \mathbb{C}^{\times}$ be the 1-dimensional representation of B^- on $\mathbb{C}$ given by

$$\rho_n\left(\begin{pmatrix} u & 0 \\ * & u^{-1} \end{pmatrix}\right) = u^n, \quad \text{where } u \in \mathbb{C}^{\times}.$$

Consider the holomorphic line bundle

$$\mathcal{L}(n) = G^{\mathbb{C}} \times_{(B^-, \rho_n)} \mathbb{C}$$

over $G^{\mathbb{C}}/B^-$. Show that the action of $G \subset G^{\mathbb{C}}$ on the space of holomorphic sections of $\mathcal{L}(n)$ defines a representation of G that is equivalent to its representation on $V(n)$.

Hint: Holomorphic sections of $\mathcal{L}(n)$ are identified with holomorphic functions $f : G^{\mathbb{C}} \to \mathbb{C}$ such that

$$f(gh) = \rho_n(h^{-1})f(g) \quad \text{for all } g \in G^{\mathbb{C}},\ h \in B^-.$$

Solution.

We note that the symmetric tensor product $V^{(n)} = S^n\mathbb{C}^2$ has the isomorphism $V^{(n)} = P_n[z_0, z_1]$, the space of homogeneous polynomials $f \in \mathbb{C}[z_0, z_1]$ of degree n, induced by $e_1 \mapsto z_0$ and $e_2 \mapsto z_1$ with $\{e_1, e_2\}$ the standard basis of $\mathbb{C}^2$. Then the representations are defined by this isomorphism.

1. The maximal torus T of G reads
$$T = \{\mathrm{diag}(u, u^{-1}) \mid u \in \mathrm{U}(1)\} \cong \mathrm{U}(1)$$
hence its Lie algebra $\mathfrak{t} \cong \mathfrak{u}(1) \cong \sqrt{-1}\mathbb{R}$ reads
$$\mathfrak{t} = \{t_\theta := \sqrt{-1}\,\mathrm{diag}(\theta, -\theta) \mid \theta \in \mathbb{R}\}.$$
The representation of $\mathfrak{t}$ on $V^{(n)}$ is explicitly, for $k = 0, \cdots, n$,
$$\begin{aligned} t_\theta \cdot (z_0^{n-k} z_1^k) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} e^{st_\theta} \cdot (z_0^{n-k} z_1^k) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (e^{s\sqrt{-1}\theta} z_0)^{n-k} (e^{-s\sqrt{-1}\theta} z_1)^k \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} e^{s(n-2k)\sqrt{-1}\theta} \cdot (z_0^{n-k} z_1^k) \\ &= \sqrt{-1}(n-2k)\theta z_0^{n-k} z_1^k. \end{aligned}$$
This means $V^{(n)}$ admits the weight space decomposition $V^{(n)} = \bigoplus_{k=0}^{n} V^{(n)}_{\lambda_k}$ with weights $\lambda_k = e^{\sqrt{-1}(n-2k)\theta}$.

2. Let $[z_0 : z_1]$ be a homogeneous coordinate on $\mathbb{C}P^1$. We define the map
$$\begin{aligned} \Phi : G^{\mathbb{C}}/B^- &\longrightarrow \mathbb{C}P^1 \\ gB^- &\longmapsto g \cdot [0 : 1] \end{aligned}$$
where the action of $G^{\mathbb{C}}$ on $\mathbb{C}P^1$ is given by the defining representation of $G^{\mathbb{C}}$, $g \cdot [z_0 : z_1] = [z_0' : z_1']$ where
$$g\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} z_0' \\ z_1' \end{pmatrix},\ g \in G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}).$$
Note that this is clearly a holomorphic action by quotient topology.

We show the map Φ is a biholomorphism: It is a holomorphic map since it descends from the holomorphic action $G^{\mathbb{C}} \longrightarrow \mathbb{C}P^1$, $g \mapsto g \cdot [0 : 1]$.

- Well-defined: For all $h \in B^-$, since $h \cdot [0 : 1] = [0 : *] = [0 : 1]$ where
$$h = \begin{pmatrix} u & 0 \\ * & u^{-1} \end{pmatrix},\ u \in \mathbb{C}^\times,$$

we have $g \cdot [0:1] = gh \cdot [0:1]$ for all $g \in G^{\mathbb{C}}$.

- 1-1 and onto: For all $[z_0 : z_1] \in \mathbb{C}P^1$ with $z_1 \neq 0$, let $g = \left(\begin{smallmatrix} z_1^{-1} & z_0 \\ 0 & z_1 \end{smallmatrix}\right)$, we have $g \cdot [0:1] = [z_0 : z_1]$. Meanwhile, let $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we have $g \cdot [0:1] = [1:0]$. Thus Φ is onto.

  On the other hand, if $g \cdot [0:1] = g' \cdot [0:1]$, say
  $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \Longrightarrow (b', d') = (\lambda b, \lambda d),$$
  for some nonzero λ. Then we have
  $$g' = g \begin{pmatrix} \lambda^{-1} & 0 \\ v & \lambda \end{pmatrix},$$
  where v is the solution of $\begin{cases} a\lambda^{-1} + bv = a' \\ c\lambda^{-1} + dv = c' \end{cases}$, which has unique solution since $\det g \neq 0$. Then we conclude $g'g^{-1} \in B^-$, hence the map Φ is 1-1.

- Finally, Φ has a holomorphic inverse given by $[z_0 : z_1] \mapsto \begin{pmatrix} a & z_0 \\ c & z_1 \end{pmatrix}$ where $az_1 - cz_0 = 1$. This is well-defined on $G^{\mathbb{C}}/B^-$ because for any solutions of $az_1 - cz_0 = 1$, say (a, c) and (a', c'), we have
  $$\begin{pmatrix} a & z_0 \\ c & z_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = \begin{pmatrix} a' & z_0 \\ c' & z_1 \end{pmatrix}, \ \begin{cases} a + z_0 v = a', \\ c + z_1 v = c', \end{cases}$$
  which has a unique solution for any fixed z_0, z_1.

3. We first inspect the structure of the line bundle

   Define the character
   $$\rho_n : B^- \longrightarrow \mathbb{C}^\times, \ \rho_n\left(\begin{pmatrix} u & 0 \\ * & u^{-1} \end{pmatrix}\right) = u^n, \ n \geq 0.$$
   Then form the associated line bundle
   $$L^{(n)} = G^{\mathbb{C}} \times_{(B^-, \rho_n)} \mathbb{C} \longrightarrow G^{\mathbb{C}}/B^- \cong \mathbb{C}P^1,$$
   where $(g, w) \sim (g\,h, \rho_n(h)^{-1}w)$ for $h \in B^-$, $w \in \mathbb{C}$.

   Local trivializations over the standard charts.

   Cover $\mathbb{C}P^1$ by the two affine charts:
   $$U_0 = \{[z_0 : z_1] : z_0 \neq 0\}, \ U_\infty = \{[z_0 : z_1] : z_1 \neq 0\}.$$
   On U_0, write $z = z_1/z_0$. Choose a holomorphic lift
   $$g_0(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in G^{\mathbb{C}},$$

which satisfies $g_0(z)\cdot[0:1]=[z:1]=[z_0:z_1]$. Define a local frame

$$s_0 : U_0 \longrightarrow L^{(n)},\ s_0\left([z_0:z_1]\right)=\left[g_0(z),\,1\right],\ z=\frac{z_1}{z_0}.$$

Any element of the fiber of $L^{(n)}$ over $[z_0:z_1]\in U_0$ can be written uniquely as $[g_0(z),w]$ for some $w\in\mathbb{C}$. Thus s_0 trivializes $L^{(n)}$ on U_0: a general section σ restricts to

$$\sigma|_{U_0}(z)=f_0(z)\,s_0(z),\ f_0(z)\in\mathcal{O}(U_0).$$

On U_∞, write $w=z_0/z_1=1/z$. Choose a holomorphic lift

$$g_\infty(w)=\begin{pmatrix}1 & 0\\ w & 1\end{pmatrix}\in G^{\mathbb{C}},$$

which satisfies $g_\infty(w)\cdot[0:1]=[w:1]=[z_0:z_1]$. Define

$$s_\infty : U_\infty \longrightarrow L^{(n)},\ s_\infty\left([z_0:z_1]\right)=\left[g_\infty(w),\,1\right],\ w=\frac{z_0}{z_1}.$$

Then any element of the fiber over $[z_0:z_1]\in U_\infty$ can be written as $[g_\infty(w),w']$ for a unique $w'\in\mathbb{C}$.

Transition function on $U_0\cap U_\infty$.

On the overlap $U_0\cap U_\infty$, both coordinates $z\neq 0$ and $w=1/z$ make sense. We need to compare the two frames s_0 and s_∞. By construction,

$$g_0(z)\cdot[0:1]=[z:1],\ g_\infty\left(z^{-1}\right)\cdot[0:1]=[z:1].$$

Hence there exists a unique

$$h(z)=\begin{pmatrix}u(z) & 0\\ v(z) & u(z)^{-1}\end{pmatrix}\in B^-$$

such that

$$g_\infty\left(z^{-1}\right)=g_0(z)\,h(z).$$

We claim $h(z)=\mathrm{diag}(z^{-1},\,z)$. Indeed,

$$g_0(z)\,\mathrm{diag}(z^{-1},z)=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}z^{-1} & 0\\ 0 & z\end{pmatrix}=\begin{pmatrix}z^{-1} & z^2\\ 0 & z\end{pmatrix},$$

while

$$g_\infty\left(z^{-1}\right)=\begin{pmatrix}1 & 0\\ z^{-1} & 1\end{pmatrix}.$$

To match these, multiply $g_0(z)\,\mathrm{diag}(z^{-1},z)$ on the right by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\in G^{\mathbb{C}}$. One finds

$$g_0(z)\,\mathrm{diag}(z^{-1},z)\,\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ z^{-1} & 1\end{pmatrix}=g_\infty\left(z^{-1}\right).$$

Since $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\in B^-$, we conclude that in the identification of frames,

$$s_\infty = \rho_n\left(\mathrm{diag}(z^{-1}, z)\right)^{-1} s_0 = z^n\, s_0 \text{ on } U_0 \cap U_\infty.$$

Indeed, $\rho_n\left(\mathrm{diag}(z^{-1}, z)\right) = (z^{-1})^n = z^{-n}$, so $\rho_n(h(z))^{-1} = z^n$. Thus the transition function of $L^{(n)}$ from U_0 to U_∞ is z^n.

Global holomorphic sections of $L^{(n)}$.

A global section $\sigma \in \Gamma\left(\mathbb{C}P^1, L^{(n)}\right)$ is given by a pair of holomorphic functions $(f_0(z),\ f_\infty(w))$ on U_0 and U_∞ satisfying the compatibility

$$f_\infty(w) = f_0(z)\ \cdot z^n \text{ whenever } w = z^{-1}.$$

Thus $f_0(z)$ must be a polynomial of degree at most n. Writing

$$f_0(z) = a_0 + a_1\, z + \cdots + a_n\, z^n,$$

one has

$$f_\infty(w) = f_0\left(w^{-1}\right)\,\left(w^{-1}\right)^n = a_0\, w^{-n} + a_1\, w^{-n+1} + \cdots + a_n,$$

which is holomorphic at $w = 0$. Hence there is a one-to-one correspondence between sections of $L^{(n)}$ and complex polynomials in z of degree $\leq n$. Equivalently, in homogeneous coordinates,

$$(a_0 + a_1\, z + \cdots + a_n\, z^n) \longleftrightarrow \sum_{k=0}^{n} a_k\, z_0^{n-k}\, z_1^k,$$

a homogeneous polynomial of total degree n. Therefore

$$\Gamma\left(\mathbb{C}P^1,\ L^{(n)}\right) \cong \{\,\text{homogeneous polynomials of degree } n \text{ in } (z_0, z_1)\} \cong \mathrm{Sym}^n$$

so $\dim \Gamma(L^{(n)}) = n + 1$.

SU(2)-equivariance.

We are left to show that the representations are equivalent.

- The action on $\mathrm{Sym}^n(\mathbb{C}^2)$ in terms of the homogeneous polynomials is explicitly as follows: Recall $\mathrm{Sym}^n(\mathbb{C}^2)$ is spanned by the monomials

$$z_0^{n-k} z_1^k,\ k = 0, \ldots, n,$$

  and for

$$g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}(2),\ |\alpha|^2 + |\beta|^2 = 1,$$

  the standard (defining) action on $\mathbb{C}^2$ is

$$(z_0, z_1) \longmapsto (z_0',\ z_1') = g \cdot (z_0, z_1) = \left(\alpha\, z_0 + \beta\, z_1, -\,\overline{\beta}\, z_0 + \overline{\alpha}\, z_1\right).$$

  Hence on homogeneous degree-n polynomials $P(z_0, z_1)$ one sets

$$\left(g \cdot P\right)(z_0, z_1) = P\left(g \cdot (z_0, z_1)\right).$$

In particular, each basis monomial transforms by

$$\circ \quad g\cdot\left(z_0^{n-k}z_1^k\right)=(\alpha\,z_0+\beta\,z_1)^{n-k}\,\left(-\overline{\beta}\,z_0+\overline{\alpha}\,z_1\right)^k.$$

The action of $\mathrm{SU}(2)$ on $\Gamma(\mathbb{C}P^1,L^{(n)})$ reads as follows: Over the standard affine chart $U_0=\{[z_0:z_1]:z_0\neq 0\}$ with $z=z_1/z_0$, every section can be written as

$$\sigma(z)=f_0(z)\,s_0(z),\;\; s_0(z)=[g_0(z),1],$$

where $g_0(z)=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$ and $f_0(z)$ is a polynomial of degree $\leq n$.

Now let

$$u=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SU}(2)\subset\mathrm{SL}(2,\mathbb{C}),\;\; ad-bc=1.$$

We wish to compute $(\,u\cdot\sigma\,)(z)\in\Gamma(L(n))$ explicitly in the chart U_0. Recall the action of $G^{\mathbb{C}}$ on $\mathbb{C}P^1$, it follows that on the affine coordinate $z=z_1/z_0$, the point $z\in U_0$ is sent (by u^{-1}) to

$$u^{-1}\cdot z=\frac{-cz+a}{dz-b}\;\text{(since } u^{-1}\cdot[z:1]=[d\,z-b:-cz+a]\,\text{)}.$$

Denote

$$z'=u^{-1}\cdot z=\frac{-cz+a}{dz-b}.$$

Recall that $s_0(z)=[g_0(z),\,1]$. Under u, this frame transforms by

$$u\cdot s_0(z)=u\cdot[g_0(z),\,1]=[u\,g_0(z),\,1]=\left[g_0\left(u^{-1}\cdot z\right)\,\cdot h(z),\,1\,\right],$$

for some $h(z)\in B^-$. Equivalently, on U_0 one finds a unique lower-triangular matrix

$$h(z)=\begin{pmatrix}u_h(z) & 0\\ v_h(z) & u_h(z)^{-1}\end{pmatrix}\in B^-$$

such that

$$ug_0(z)=g_0\left(u^{-1}\cdot z\right)\,h(z).$$

A direct computation shows

$$g_0(z)=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix},\;u\,g_0(z)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a & a\,z+b\\ c & cz+d\end{pmatrix}.$$

On the other hand,

$$g_0(z')=\begin{pmatrix}1 & z'\\ 0 & 1\end{pmatrix},\;z'=\frac{-cz+a}{dz-b}.$$

Hence

$$g_0(z')\,h(z)=\begin{pmatrix}1 & z'\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_h & 0\\ v_h & u_h^{-1}\end{pmatrix}=\begin{pmatrix}u_h+z'\,v_h & z'\,u_h^{-1}\\ v_h & u_h^{-1}\end{pmatrix}.$$

We must have

$$u\, g_0(z) = g_0(z')\, h(z),$$

i.e.

$$\begin{pmatrix} a & a\,z+b \\ c & cz+d \end{pmatrix} = \begin{pmatrix} u_h + z'\,v_h & z'\,u_h^{-1} \\ v_h & u_h^{-1} \end{pmatrix}.$$

Equating entries yields

$$\begin{cases} v_h = c, \\ u_h^{-1} = cz+d, \\ u_h + z'\,v_h = a, \\ z'\,u_h^{-1} = a\,z+b. \end{cases}$$

From $v_h = c$ and $u_h^{-1} = cz+d$, we get

$$u_h = \frac{1}{cz+d},\ \ v_h = c.$$

One checks consistency with $z' = (-cz+a)/(dz-b)$ and $z'\,u_h^{-1} = a\,z+b$. Thus

$$h(z) = \begin{pmatrix} (cz+d)^{-1} & 0 \\ c & cz+d \end{pmatrix} \in B^-.$$

In particular,

$$u_h(z) = \frac{1}{cz+d},\ \ v_h(z) = c.$$

Since $s_0(z) = [g_0(z), 1]$, it follows that

$$u \cdot s_0(z) = [g_0(z')\,h(z),\ 1] = \left[g_0(z'),\ \rho_n\,(h(z))^{-1}\right] = \rho_n\,(h(z))^{-1}\ s_0(z'),$$

where

$$\rho_n\,(h(z)) = (\,u_h(z))^n = (cz+d)^{-n}\,.$$

Hence

$$u \cdot s_0(z) = (cz+d)^n\, s_0\,(z')\,,\ \ z' = u^{-1}\cdot z = \frac{-cz+a}{dz-b}.$$

paragraph4. Action on a general section $\sigma(z) = f_0(z)\,s_0(z)$. Given $\sigma(z) = f_0(z)\,s_0(z)$, the SU(2)-action is

$$(u\cdot\sigma)(z) = u\cdot(f_0(z)\,s_0(z)) = f_0\left(u^{-1}\cdot z\right)(u\cdot s_0(z))\,.$$

Substitute the formula for $u \cdot s_0(z)$:

$$u \cdot s_0(z) = (cz+d)^n\ s_0\,(z')\,,\ \ z' = \frac{-cz+a}{dz-b}.$$

Thus

$$(u \cdot \sigma)(z) = f_0\left(z'\right)(cz+d)^n\, s_0\left(z'\right).$$

But $s_0(z')$ is merely the local frame at the new point z'. Hence the new section in the U_0-trivialization is

$$(u \cdot \sigma)_0(z) = (cz+d)^n\, f_0\left(\frac{-cz+a}{dz-b}\right).$$

In other words, if σ corresponds (in U_0) to the polynomial $f_0(z)$, then

$$(u \cdot f_0)(z) = (cz+d)^n f_0\left(\frac{-cz+a}{dz-b}\right), \; u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Equivalently, in homogeneous coordinate form, $u \cdot P$ on $P(z_0, z_1)$ is

$$(u \cdot P)\left(z_0, z_1\right) = P\left(\left(z_0, z_1\right) u\right) = P\left(a\, z_0 + b\, z_1, cz_0 + d\, z_1\right), \; u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These two descriptions agree under the inhomogeneous identification $P(z_0, z_1)|_{z_0=1} = f_0(z_1)$, since

$$P\left(a + b\, z, c + d\, z\right) = (cz+d)^n P\left(\frac{a + b\, z}{cz+d}, 1\right).$$

We conclude they are indeed isomorphic representations.

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1. By Jacobi identity,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \Longrightarrow [a, [b, c]] - [b, [a, c]] = [[a, b], c].$$

↩