Correcting Gradshteyn and Ryzhik, Integral 3.411.16

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Note: I previously posted a wrong version of this, where I messed up the calculus and came to a wrong conclusion. This is the corrected version.

I stumbled across the following delightfully weird result in Gradshteyn and Ryzhik, Table of Integrals, Series, and Products:

$$\int_{-\infty}^{\infty} \frac{x^2 e^{-\mu x}}{1+e^{-x}} dx = \pi^3 \cos^3 \mu\pi (2 - \sin^2 \mu\pi)$$

(for $0 < \text{Re } \mu < 1$)

This is entry 3.411.16. It's been there for quite some time (I originally found it in the sixth edition, which is the version I have and comes from 1969, but it's still present in the 8th edition). It originally comes from Tables of Integral Transforms, from the Harry Bateman project, edited by Erdélyi.

Unfortunately, the reason it's so delightfully weird is that it's wrong. The version it's citing is correct, but it's made two errors in transcription:

1. It is a transcription error from the original, which has csc instead of cos
2. It dropped some brackets from the original, making it look like the $\sin^2$ was inside the $\cos^3$ call.

It's possible that the intended reading of this *is* the multiplicative form and I'm just misreading it, but the replacement of the csc with the cos is definitely a genuine error.

The correct form of this integral is:

$$\int_{-\infty}^{\infty} \frac{x^2 e^{-\mu x}}{1+e^{-x}} dx = \pi^3 \csc^3(\mu\pi)(2 - \sin^2 \mu\pi)$$

We can derive the correct answer easily, starting from the following result (which I'll prove in a moment):

$$\int_{-\infty}^{\infty} \frac{e^{-\mu x}}{1+e^{-x}} dx = \frac{\pi}{\sin \pi\mu}$$

From here, differentiating twice, we get:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2 e^{-\mu x}}{1+e^{-x}} dx &= \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} \int_{-\infty}^{\infty} \frac{e^{-\mu x}}{1+e^{-x}} dx \\
&= \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} \frac{\pi}{\sin \pi\mu} \\
&= -\pi^2 \frac{\mathrm{d}}{\mathrm{d}\mu} \frac{\cos \pi\mu}{\sin^2 \pi\mu} \\
&= -\pi^3 \left[\frac{\sin \pi\mu}{\sin^2 \pi\mu} - 2 \frac{\cos^2 \pi\mu}{\sin^3 \pi\mu} \right] \\
&= -\pi^3 \csc^3(\pi\mu) \left[-\sin^2(\pi\mu) - 2\cos^2(\pi\mu) \right] \\
&= \pi^3 \csc^3(\pi\mu) \left[\sin^2(\pi\mu) + 2\cos^2(\pi\mu) \right] \\
&= \pi^3 \csc^3(\pi\mu) \left[2 - \sin^2(\pi\mu) \right]
\end{aligned}$$

Which is the correct form of the result, and the version originally in “Tables of Integral Transforms”.[1]

Now, let’s calculate the integral we used:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{-\mu x}}{1+e^{-x}} dx &= \int_{-\infty}^{0} \frac{e^{-\mu x}}{1+e^{-x}} dx + \int_{0}^{\infty} \frac{e^{-\mu x}}{1+e^{-x}} dx \\
&= \int_{0}^{\infty} \frac{e^{\mu x}}{1+e^{x}} dx + \int_{0}^{\infty} \frac{e^{-\mu x}}{1+e^{-x}} dx \\
&= \int_{0}^{\infty} \frac{e^{-(1-\mu)x}}{1+e^{-x}} dx + \int_{0}^{\infty} \frac{e^{-\mu x}}{1+e^{-x}} dx \\
&= \beta(1-\mu) + \beta(\mu)
\end{aligned}$$

[1]Initially I thought that the version from Tables of Integral Transforms was also wrong, but this was a misreading that I was primed to by the dropped brackets.

Where $\beta(x) = \frac{1}{2}\left[\psi(\frac{x+1}{2}) - \psi(\frac{x}{2})\right]$, and ψ is the digamma function.

This result is 8.371.2 from Table of Integrals, Series, and Products. I originally worked it out myself, which you can do using $\frac{1}{1+e^{-x}} = \sum_{n\geq 0} e^{-nx}$ and some algebraic manipulation and standard results. I omit the proof here.

We can use the reflection formula for the digamma function, that $\psi(1-x)-\psi(x) = \pi\cot(\pi x)$, to get a similar nice reflection formula for β, which will give us our desired integral.

$$\begin{aligned}
f(\mu) &= \beta(1-\mu) + \beta(\mu) \\
&= \frac{1}{2}\left[\psi(\frac{(1-\mu)+1}{2}) - \psi(\frac{1-\mu}{2}) + \psi(\frac{\mu+1}{2}) - \psi(\frac{\mu}{2})\right] \\
&= \frac{1}{2}\left[\psi(1-\frac{\mu}{2}) - \psi(\frac{1-\mu}{2}) + \psi(1-\frac{1-\mu}{2}) - \psi(\frac{\mu}{2})\right] \\
&= \frac{1}{2}\left[\left(\psi(1-\frac{\mu}{2}) - \psi(\frac{\mu}{2})\right) + \left(\psi(1-\frac{1-\mu}{2}) - \psi(\frac{1-\mu}{2})\right)\right] \\
&= \frac{\pi}{2}\left[\cot(\frac{\mu\pi}{2}) + \cot(\frac{\pi}{2} - \frac{\mu\pi}{2})\right] \\
&= \frac{\pi}{2}\left[\cot(\frac{\mu\pi}{2}) + \tan(\frac{\mu\pi}{2})\right] \\
&= \frac{\pi}{\sin\pi\mu}
\end{aligned}$$

Where the last line uses the identity that $\tan(x) + \cot(x) = \frac{2}{\sin 2x}$.

Some sanity checks: This tends to infinity at 0 and 1 like it should, and is symmetric around $\mu = \frac{1}{2}$ like it should be. I've also plugged in a few special cases of μ and checked numerically that the answer is correct. So I'm now pretty confident in this result.