An easy to remember proof of trigonometric sum identities

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It's been a remarkably long time since I've needed to remember trigonometric addition identities, and I found myself lying in bed awake trying to remember a) what they were and b) how to prove them. I could, of course, have gotten up and looked them up, but that would have been admitting defeat.

I'm aware there's a neat geometric proof, but the chances of my remembering that or being able to do it in my head round to zero.[1]

My first thought was to just brute force it with the identities $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2}$, but that seemed gross and doing that algebra in my head seemed fraught.

But then I hit upon the following fact: The trigonometric functions are essentially defiend by their ODE. Every solution to $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -y$ is a linear combination of $\cos$ and $\sin$.

From there, it's easy. $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(a+x) = -\cos(a+x)$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(a+x) = -\sin(a+x)$, so these both must be linear combinations of $\cos$ and $\sin$. You can tell which ones by plugging in values. $\cos(a+x) = A\cos(x) + B\sin(x)$. By evaluating at $x = 0$, you know that $A = \cos(A)$. By taking the derivative to get $\frac{\mathrm{d}}{\mathrm{d}x}\cos(a+x) = -\sin(a+x)$ you know that $B = -\sin(A)$. So $\cos(a+x) = \cos(a)\cos(x) - \sin(a)\sin(x)$ as desired.

You can apply essentially the same reasoning to get $\sin(a+x) = \sin(a)\cos(x) + \cos(a)\sin(x)$.

I find it very useful to have proofs that can fit in my head, because they're great for being able to remember things. Rather than treating mathematical facts in isolation, they all fit together and as long as I remember one piece I can follow the links to get back to the result I want.

1. I have a fairly large degree of aphantasia so am essentially never able to do geometry in my head.↩