Classical Satake

J'ignore • 25 Mar 2026

We have the Chevalley restriction theorem:

The restriction of the natural map from the ring of polynomial class functions on the lie algebra $\mathfrak{g}$ to the ring of polynomial functions on $\mathfrak{t}$ is injective, and its image coincides with the invariant polynomial ring with respect to the action of the Weyl group W on $\mathfrak{t}$.

Note that $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathfrak{t}^*$, so $\mathbb{C}[X^*(T)]$, the ring of polynomial functions on $X^*(T)$, is identified with $\mathbb{C}[\mathfrak{t}^*]$, which is the ring of polynomial functions on $\mathfrak{t}$.

We further have an isomorphism of algebras from $\mathbb{C} \otimes_{\mathbb{Z}} K(Rep_{G^\vee})$ to the algebra of polynomial class functions on $G^\vee$ by taking character.

The Bruhat decomposition tells us that the double coset algebra $\mathbb{C}[B \setminus G/B]$ is isomorphic to $\mathbb{C}[W]$ (This is the specialization of the one-parameter family of Iwahori-Hecke algebras $\mathcal{H}_q$ specialized at $q = 1$; for all other values of q except at roots of unity, Tits prove it is isomorphic to it is also isomorphic to $\mathbb{C}[W]$; See here, point 4). We would like to obtain similar description for maximal compact K. See the wiki article. This will give us the unramified local Langlands correspondence, where the parameter is just a point of $\widehat{T}(\mathbb{C})/W$, or a semisimple element of $\widehat{G}(\mathbb{C})$ up to conjugacy. see here for a sketch of the proof. The key is Iwasawa's decomposition $G = BK$.

Proof that Sf is W-invariant (note that it is important to put in the $\delta^{1/2}$ factor (see page 147 of Cartier's article).

Reference:

https://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/L4.pdf

https://kentajsuzuki.github.io/seminar-talk/yuta-talk.pdf

https://people.math.harvard.edu/~gross/preprints/sat.pdf

own note