Getz's automorphic form chapter 8

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See Prop. 8.2.2. for the definition of parabolic induction preserves admissibility (using Iwasawa decomposition) and unitarizability (motivation for the modular character fudge factor). We also introduce a variant $I(\varphi, \lambda)$ to make clear that the induced representation $I(\sigma, \lambda)$ is part of a continuous family of representations indexed by $\mathfrak{a}^*_{M\mathbb{C}} = X^*$ where the normalized induction functor corresponds to ρ half-sum of positive root. The Jacquet module V_N is a left adjoint to parabolic induction and it also preserves admissibility.

The key to the equivalence between supercuspidality is Renard's note III.1.5. The proof that irreducible supercuspidal representations are admissible also uses it, which then implies irreducible smooth representations are admissible. From the Frobenius reciprocity we can attach cuspidal support (M, W) to an irreducible smooth representation V, i.e. one in which there is an intertwining map from V to $Ind_M^P W$ for some parabolic P containing M.

We have the nonarchimedean analogue of Langlands classification, which states that every admissible representation arises as irreducible quotient of parabolic induction of some Langlands data consisting of a tempered representation of Levi M and $\lambda \in \mathfrak{a}^*_{M\mathbb{C}}$. To classify tempered representation of GL_n, we further have the Bernstein-Zelevinsky classification. First we can build all essentially square integrable representations from supercuspidal representations by parabolic induction from $GL_{n/a}(F)$ and taking irreducible quotients $Q(\sigma^a, \lambda_a)$. If it has unitary central character, then it is square integrable. Then we can build tempered representations by taking parabolic induction from $GL_{n_1}(F) \times ... \times GL_{n_r}(F)$ of these $Q(\sigma_i^{a_i}, \lambda_i)$, which will be irreducible and temepred if all the Q are square integrable and no two of them are linked.

Motivation for generic representation: If σ representation of $GL_2(k)$ for a finite field k doesn't admit a nonzero vector fixed by the unipotent radical $N(k)$, then the compact induction $c - ind_{ZK}^G \sigma$ is an irreducible supercuspidal representation of $GL_2(F)$.

Motivation for Weil-Deligne representation: to account for Steinberg representation arising from reducible principal series (limit of discrete series representations).

Further explanation:

Regarding (1), from the point of view of Galois representations, the point is that continuous Weil group representations on a complex vector space, by their nature, have finite image on inertia.

On the other hand, while a continuous ℓ-adic Galois representation of $G_{\mathbb{Q}_p}$ (with $\ell \neq p$ of course) must have finite image on wild inertia, it can have infinite image on tame inertia. The formalism of Weil–Deligne representations extracts out this possibly infinite image, and encodes it as a nilpotent operator (something that is algebraic, and doesn't refer to the ℓ-adic topology, and hence has a chance to be independent of ℓ).

As for (2): Representations of the Weil group are essentially the same thing as representations of $G_{\mathbb{Q}}$ which, when restricted to some open subgroup, become abelian. Thus (as one example) if E is an elliptic curve over $\mathbb{Q}$ that is not CM, its ℓ-adic Tate module cannot be explained by a representation of the Weil group (or any simple modification thereof). Thus neither can the weight 2 modular form to which it corresponds.

In summary: the difference between the global and local situations is that an ℓ-adic representation of $G_{\mathbb{Q}_p}$ (or of G_E for any p-adic local field) becomes, after a finite base-change to kill off the action of wild inertia, a tamely ramified representation, which can then be described by two matrices, the image of a lift of Frobenius and the image of a generator of tame inertia, satisfying a simple commutation relation. On the other hand, global Galois representations arising from ℓ-adic cohomology of varieties over number fields are much more profoundly non-abelian.

Reference on Jacquet functor: https://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/L3.pdf

Reference on supercuspidal represetnation: https://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/L5.pdf