Springer theory intro

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The goal is to construct irreducible representation of Weyl group in a geometric way rather than combinatorial. Recall the Springer resolution $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ of the nilpotent cone $\mathcal{N}$ where $\tilde{\mathcal{N}}$ can also be identified with the cotangent bundle of the flag variety $T^*\mathcal{B}$.

The nilpotent cone $\mathcal{N}$ are acted on by $G \times \mathbb{G}_m$ where $\mathbb{G}_m$ acts by dilation. If we let $\tilde{G}_e$ be the stablizer of a nilpotent element e, then $\tilde{G}_e$ acts on the Springer fiber $\mathcal{B}_e$. If we consider the induced action on the cohomology $H^*(\mathcal{B}_e)$, then the action of $\tilde{G}_e$ factors through the component group $\pi_0(\tilde{G}_e)$. Since $\tilde{G}_e$ surjects onto $\mathbb{G}_m$ with kernel $C_G(e)$, the component group $A_e := \pi_0(C_G(e))$ surjects onto $\pi_0(\tilde{G}_e)$, so we get a representation of A_e.

Let W be the Weyl group of G. Springer discovered that even though W doesn't act directly on the Springer fiber $\mathcal{B}_e$, there is a natural W-action on the cohomology $H^*(\mathcal{B}_e)$. Moreover, this action commute with the above action by A_e. For every $e \in \mathcal{N}$ and every irrep ρ of A_e, the ρ-isotypic component $M(e, \rho)$ will exhaust all the irrep of the Weyl group W. See Thm. 1.5.1 of Yun's note.

For example, if $e = 0$, then the Springer representation can be identified with the induced action on $H^*(G/T) \cong H^*(\mathcal{B})$ from the right action of W on G/T. Using Borel's presentation of the cohomology ring of the flag and Chevalley's restriction theorem, we can further identify this representation with the regular representation of W.

One construction is by the theory of perverse sheaves. The key theorem is Thm. 1.5.7, which says the complex $\mathcal{S} := R\pi_*\mathbb{Q}_\ell[\dim \mathcal{N}]$ is a perverse sheaf on $\mathcal{N}$ whose endomorphism ring is canonically isomorphic to the group ring $\mathbb{Q}_\ell[W]$. In particular, W acts on the stalk of $R\pi_*\mathbb{Q}_\ell$, i.e. it acts on $H^*(\mathcal{B}_e)$ for all $e \in \mathcal{N}$. The idea is that by the dimension formula for the Springer fiber, one sees that the Springer resolution is semismall and the Grothendieck resolution $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is small, which is a technical condition that guarantees $\mathcal{S}_{\mathfrak{g}}$ is perverse and it is the intermediate extension of its restriction to any open dense subset of $\mathfrak{g}$. In particular over the regular semisimple locus $\pi_{\mathfrak{g}}$ is a W-torsor and $\mathcal{S}_{\mathfrak{g}}|_{\mathfrak{g}^{rs}}$ is a W-local system placed in dimension $-\dim \mathfrak{g}$ By the functoriality of intermediate extension we get an action of W on $\mathcal{S}_{\mathfrak{g}}$.

There is another proof using the Steinberg variety $Z := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. Using this Kazhdan-Lusztig give a topological construction in this paper. It is the union of the conormal bundles to all G-orbits in $\mathcal{B} \times \mathcal{B}$, which are indexed by the Weyl group W, and the irreducible components of Z are the closures of $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$. See Ginzburg's book Corollary 3.3.5. As in the above proof a key role is the following dimension identity: Let $\mathbb{O}_e$ be the conjugacy class of $e \in G$. Then we have $\dim \mathcal{B}_e + \frac{1}{2} \dim \mathbb{O}_e = \dim \mathcal{B}$. See Corollary 3.3.24. The proof uses some symplectic geometry and it still to be understood.

Lie algebraic version of the story: The exponential map is G-equivariant and identifies $\mathcal{N}$ with $\mathcal{U}$ for $char(k)$ large.

Reducedness of Springer fiber:

For $G = SL_2$, the fiber $\tilde{\mathcal{B}}_e$ is non-reduced. We can compute this by parametrizing Borels by $[x : y]$, and the Lie algebra of the corresponding Borel can be identified with subsets of $\mathfrak{sl}_2 = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}$ that fixed the line defined by $[x : y]$. Then the equation for $\tilde{B}_e$ is $y^2 = 0$. On the other hand, the equation for $\mathcal{B}_e$ is reduced, as its equation becomes $y = 0$.

Example:

For $G = SL_n$, the Springer correspondence is supposed to give us a bijection between irreps of S_n indexed by Young Tableaux λ (the corresponding representation S_λ has basis in bijection with the set of standard Young tableaux of shape λ) and nilpotent conjugacy classes indexed by partitions of n. Say the top cohomology of the Springer fiber $\mathbb{B}_e$ realizes this representation for e in the nilpotent conjugacy class $\mathcal{O}$. The nilpotent Steinberg variety Z has irreducible components indexed by the Weyl group W. On the other hand, we can also describe the cohomology $H(Z)$ as direct sum of $H_{\mathcal{O}}$ where $\mathcal{O}$ range over nilpotent orbits (to show this we first define a filtration of Z by $Z_{\leq \mathcal{O}}$; then we show $H(Z_{\leq \mathcal{O}})$ is a two-sided ideal in $H(Z)$. The key is to show $H(Z)$ is semisimple.). Each $H(\mathcal{O})$ is isomorphic to $(H(\mathcal{B}_x) \otimes H(\mathcal{B}_x))^{C(x)}$. Finally, we deduce $H(Z) \cong \bigoplus_{x, \chi \in C(x)^\wedge} \mathrm{End}_{\mathbb{C}}(H(\mathcal{B}_x)_\chi)$. Since each $H(\mathcal{B}(x))$ has a vector space basis that can be identified with the set of standard Young tableaux of shape λ, we get a bijection between the set of pairs of standard Young tableaux of the same shape and S_n.

Reference:

Yun's note on Springer theory

Ginzburg's book

own note