Chapter 2 of Mumford's Abelian Variety

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1. Abelian variety is commutative. The classical proof for $k = \mathbb{C}$ is based on considering the adjoint representation of X in the tangent space at e. In the general case we consider $C_{e,n} := \mathcal{O}_e/\mathfrak{m}^n$. Consider $Ad_x : X \to X$. This induce an automorphism of the vector space $C_{e,n}$. Thus we obtain a set-theoretic map $X \to Aut(C_{e,n})$. If we put on the target the natural structure of an algebraic variety over k, then we can check this is a morphism of varieties. Since the latter is affine and X is complete and connected, this map must be constant! Since $\cap_n \mathfrak{m}^n = 0$ this means that $Ad_x^* : \mathcal{O}_e \to \mathcal{O}_e$ is identity so Ad_x reduce to the identity in a neighborhood of e in X. Since X is irreducible, we are done.

2. The following is true for general group variety: We have an isomorphism of sheaves $\Omega_0 \otimes_k \mathcal{O}_X \cong \Omega_X^1$ where $\Omega_0 := T_{X,0}^*$. Since X is complete, $H^0(X, \mathcal{O}_X) = k$, so everywhere regular forms on X are precisely the invariant forms.

3. We can show $n_X : X \to X$ given by $x \mapsto x + x + ... + x$ is surjective for $n \nmid char(k)$ by inspecting it on $T_{X,e}$ and using the dimension formula.

4. Rigidity lemma: If X complete variety and $f : X \times Y \to Z$ is a morphism of varieties such that $f(X \times \{y_0\})$ is mapped to a single point z_0, then there exists $g : Y \to Z$ such that $f = g \circ pr_2$.

Idea: Define g by choosing any $x_0 \in X$ and $y \mapsto f(x_0, y)$. By the completeness of X, for any affine open neighborhood of U of z_0, we have $G := pr_2(f^{-1}(Z \setminus U))$ is closed in Y and $V := Y \setminus G$ is nonempty oopen (since $y_0 \in V$) and satisfies the requirement that $X \times \{y\}$ gets mapped by f into the affine U, hence is mapped to a single point.

5. Corollary: Any morphism between abelian variety is a group homomorphism up to translation.

6. The functor $S \mapsto Hom(S, X)$ is linear on the category of complete varieties with base point.

7. There is even a weaker characterization of abelian varieties: A morphism $m : X \times X \to X$ on a complete variety X with a point $e \in X$ satisfying $m|_{X \times \{e\}} = m|_{\{e\} \times X} = id_X$ makes X an abelian variety. See page 44-45.

8. Any proper morphism of Noetherian schemes $f : X \to Y$ with $Y = Spec(A)$ affine, $\mathcal{F}$ coherent sheaf on X, flat over Y. There is a finite complex of f.g. projective A-modules $K^\bullet$ and an isomorphism of functors: $H^p(X \times_Y Spec(B), \mathcal{F} \otimes_A Spec(B)) \cong H^p(K^\bullet \otimes_A B)$ on the category of A-algebra B.

9. Corollary 1: The dimension of cohomology groups $y \mapsto \dim_{\kappa(y)} H^p(X_y, \mathcal{F}_y)$ is upper semicontinuous on Y and the euler characteristic $y \mapsto \chi(\mathcal{F}_y)$ is locally constant on Y.

10. Corollary 2: We can derive the proper base change for coherent sheaves, and it also follows that if the dimension of $H^i(X_y, \mathcal{F}_y)$ is constant in y (so $R^i f_* \mathcal{F}$ is locally free sheaf and base change holds) then base change holds for $R^{i-1}\mathcal{F}$ (by right exactness of $\otimes$).

11. Seesaw theorem (family of line bundles on complete varieties). We use here the homological criterion for triviality of a line bundle and the upper semicontinuity.

12. Corollary: Theorem of the cube (it more or less says that $Pic(X) = H^1(X, \mathcal{O}^\times)$ is a quadratic functor, which is obvious from the exponential exact sequence if $k = \mathbb{C}$, since Kunneth implies that $H^2(X, \mathbb{Z})$ is of order 2 and $H^1(X, \mathcal{O})$ is of order 1 when X is complete so $H^0(X, \mathcal{O}) = k$). Morally the quadratic nature of Pic follows from that of the component group, which is naturally subgroup of $H^2(X, \mathbb{Z})$.

13. Corollary: If $\mathcal{L}$ is a line bundle on $X \times Y \times Z$, then there is $\mathcal{L}_i$ for $i = 1, 2, 3$ such that $\mathcal{L}$ is tensor product of pullback of $\mathcal{L}_i$ under projection $p_{\widehat{i}}$.

14. Corollary: computation of $(f + g + h)^*\mathcal{L}$ (use Theorem of the cube)

15. Corollary 1 of (14): computation of $n^*\mathcal{L}$ ((14) implies the second difference $(n+2)^*\mathcal{L} \otimes (n+1)^*\mathcal{L}^{-2} \otimes n^*\mathcal{L}$ is independent of n and isomorphic to $1^*\mathcal{L} \otimes (-1)^*\mathcal{L}$.

16. Corollary 2 of (14): Theoerem of square (For any line bundle $\mathcal{L}$ we have the Abel Jacobi map $\phi_{\mathcal{L}} : X \to Pic(X)$ given by $x \mapsto T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, the fudge factor $\mathcal{L}^{-1}$ is because $T_x^{\mathcal{L}}$ is second order)

17. A generic effective divisor $\mathcal{L} := \mathcal{L}(D)$ on X is ample (in the sense that the closed subgroup $H := Stab(\mathcal{L})$ is finite). This implies that abelian varieties are projective.