Equivariant derived categories

J'ignore • 9 Feb 2026

A principal G-bundle $\pi : X \to Y$ need not be Zariski locally trivial, e.g. see here, but it is at least etale-locally trivial. The idea is that every point $y \in Y$ admits an etale neighborhood over which π has a right inverse.

A stronger claim is true if G is affine: Every principal bundle is locally isotrivial. This means the etale neighborhood can be chosen to be finite over its image in Y (i.e. the number of points in the fiber is constant, see here), e.g. see this post.

An important proposition: Let P be a principal G-variety, and X another G-variety, then $P \times X$ is a principal G-variety. For this we need X to be quasiprojective as well to ensure (*) Every finite subset of X is contained in an affine open subset.

A G-equivariant sheaf on a G-variety X is a sheaf $\mathcal{F}$ together with a rule θ that assigns to every $g \in G$ and U open an isomorphism $\theta_{g,U} : \mathcal{F}(U) \to \mathcal{F}(g \cdot U)$, and it should satisfies some associativity. Equivalently, we can say for every $g \in G$, there is a map $\theta_g : \mathcal{F} \to \sigma_g^*(\mathcal{F})$ satisfying associativity where σ_g is the action of g on X. However, this doesn't take into account the topology of G (in other words we just treat G like a discrete group). A better definition is to view θ_g as part of $\theta : pr_2^*\mathcal{F} \to \sigma^*\mathcal{F}$ in $Sh(G \times X, R)$. Then the associativity axiom is $b^*\theta \circ pr_{23}^*\theta = m^*\theta$ where $m : G \times G \times X \to G \times X$ is multiplication of the first two coordinates, $b : G \times G \times X \to G \times X$ is the action of the second coordinates, and $pr_{23} : G \times G \times X$ is the projection onto the last two coordinates. The idea is that $b^*\theta|_{\{g\}\times\{h\}\times X = \sigma_h^*\theta_g}$, $\theta_h = pr_{23}^*\theta|_{\{g\}\times\{h\}\times X}$, and $\theta_{gh} = m^*\theta|_{\{g\}\times\{h\}\times X}$, so this boils down to the usual axiom. The Proposition is that $Sh_G(X, R)$ is an abelian category.

There are certain subtlety relating to equivariant derived category. It is neither $D_c^b Perv_G$ nor just copy the above definition but allowing $\mathcal{F}$ to be a general object in D_c^b. See section 6.4. We want the definition to satisfy the desideratum 6.1, including forgetful functor and six-functor formalism that is mutually compatible. See example 6.4.2 that uses the below proposition.

(Prop. 6.2.13) Every G-equivariant perverse sheaf on a G-homogeneous space X is a shifted local system and there is an equivalence of categories $Loc_G^{ft}(X, k) \cong k[G_x/(G_x)^\circ]\text{-mod}^{fg}$.

However, if X is a principal G-variety with geometric quotient $\pi : X \to X/G$, then we can define $D^b_G(X, k)$ as $D^b_c(X/G, k)$. The key is Proposition 6.2.10 asserting $\pi^\dagger$ induces an equivalence of categories between $Perv(X/G)$ and $Perv_G$. This is just smooth descent for perverse sheaves plus identifying descent datum with the definition of G-equivariant sheaves using the Cartesian diagram for principal G-varieties.

For general G-varieties we use acyclic G-resolutions. See Exercise 3.7.2 for the definition of n-acyclic morphism. The key point is that for n-acyclic smooth morphism $f : X \to Y$ the shifted pullback $f^\dagger : {}^pD^b_c(Y, k)^{[k,k+n]} \to {}^pD^b_c(X, k)^{[k,k+n]}$ is fully faithful (easily seen from the adjunction). The motivation is that the cohomology of the fibers of f can lead to 'extra' morphisms in $D^b_c(X, k)$ or in the category of descent data so there is no smooth descent on the level of derived category. However, if f is ∞-acyclic, then for all $\mathcal{F} \in D^b_c(Y, k)$, the unit and counit map $\mathcal{F} \to f_*f^*\mathcal{F}$ and $f_!f^!\mathcal{F}$ are both isomorphisms.

Lemma 6.1.20 tells us when $X \to pt$ is an n-acyclic morphism in terms of $H^i(X; \mathbb{Z})$. The idea is to reduce to the case of constant sheaf $\underline{\mathbb{Z}}$ on X and apply Exercise 3.2.1 (t-exactness of $\otimes^L$ under good situations). From this we can produce lots of n-ayclic $\mathbb{G}_m$ resolutions and GL_k-resolutions more generally. See Example 6.1.21 and 6.1.22. For linear algebraic group G this is suffice to conclude the existence of n-acyclic G-resolution of any G-variety X using Prop. 6.1.13 and Prop. 6.1.15.

Now the definition for $D^b_G(X, k)$ as in Definition 6.4.4 appears very natural. Essentially for every G-resolution $P \to X$ we give an object $\mathcal{F}(p) \in D^b_c(P/G, k)$ and it should be compatible for different G-resolutions in the sense that if $\nu : P \to Q$ is a morphism of G-resolution then there is an isomorphism $\alpha_\nu : \overline{\nu}^*\mathcal{F}(q) \to \mathcal{F}(p)$. Moreover, the isomorphism α_ν should compose.

Note that we always have the trivial G-resolution $pr_2 : G \times X \to X$ which allows us to define the forgetful functor $\mathcal{F} \mapsto \mathcal{F}(pr_2)$. It is not evidently a triangulated category with heart $Perv_G$. See Berstein-Lunt's book for detailed proof.

Reference: Achar's book