Reference for local langlands

J'ignore   •   17 Jan 2026

Main reference: Oliver Taibi's note

Characterization of supercuspidal representation in terms of matrix coefficients: Theorem 5.3.1 of Casselman's book

Langlands classficiation: classify irreducible representations of a reductive Lie group G in terms of tempered representations of smaller groups. Since tempered representations are in turn given as certain representations induced from discrete series or limit of discrete series representations, one can do both inductions at once and get a Langlands classification parameterized by discrete series or limit of discrete series representations instead of tempered representations. The problem with doing this is that it is tricky to decide when two irreducible representations are the same.

Different decompositions of p-adic groups:

Cartan decompsition of Lie algebra of a real Lie group, generalizing the polar decomposition: $G = KAK$

Iwasawa decomposition of Lie algebra of a real semisimple Lie group or over a non-Archimedean field, generalizing the Gram–Schmidt orthogonalization: $G = KAN$

Langlands decomposition: $L = MA$ where $M \subset K$ reductive and A is the split center of L. The unique minimal parabolic (Borel) is $P = LN = MAN$.

Representation of SL_2: Using the fact that it is an eigenvector of the Casimir operator and has an eigenvector for H, it follows easily that any irreducible admissible representation is a subrepresentation of a parabolically induced representation. (This also is true for more general reductive Lie groups and is known as Casselman's subrepresentation theorem.) The irreducible unitary representations can be found by checking which of the irreducible admissible representations admit an invariant positively definite Hermitian form.

Reference on real group representations

Classical fact: For $G = GL_n$, any smooth representation $G(F)$ is either one-dimensional or infinite dimensional (the kernel of the representation is open normal; and it contains the standard unipotent subgroup so it contains SL_n; this argument fails for $D^\times$ for a quaternion algebra D since $D^\times$ has no unipotent and $D^\times(F)$ is compact).

Langlands dual group:

Let G be a connected reductive group over F. Take a finite separable extension E/F s.t. G_E admits a Killing/Borel pair (B, T). Let $(X, R, R^\vee, \Delta)$ be the associated (reduced) based root datum. Here X is the group of characters, $R \subset X$ is the set of roots of T in G_E, $R^\vee$ is the set of coroots, which is a subset of $X^\vee$ and $\Delta \subset R$ is a set of simple roots (we should also include a bijection $R \leftrightarrow R^\vee$). All other choices of Killing pair in G_E yield based root data canonically isomorphic to $(X, R, R^\vee, \Delta)$, and so do other choices for E.

There is also a continuous action of Γ on this based root datum. First Γ acts on the set of closed subgroups of G, and there is a unique $T(E)g_\sigma \in T(E) \backslash G(E)$ s.t. $\sigma(B, T) = Ad(g_\sigma)^{-1}(B, T)$. Then we define $\sigma(\lambda)$ where $\lambda \in X$ to be $\sigma(\lambda) \circ Ad(g_\sigma)$.

Inner twist/forms of a reductive group G over F:

A reductive group G' over F is an inner form of G if there is an isomorphism $\phi : G_{\overline{F}} \to G_{\overline{F}}$ such that for any $\sigma \in \Gamma$, we have $\phi^{-1} \circ \phi^\sigma$ is inner.

We can 'classify' connected reductive groups over F as follows: - Fix a representative in each isomorphism class of based root datum with continuous action of Γ;

- For each such representative b, fix a quasi-split connected reductive group G^* over F together with an isomorphism $brd_F(G^*) \cong b$;

- For each element of $H^1(F, G^*_{ad})$ choose an inner twist (G, ψ) of G^* representing it.

Construction of $\widehat{G}$ and $^L G$:

The choice of a pinning of $\widehat{G}$ induces a splitting of the extension

$$1 \to \widehat{G}_{ad} \to \mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{G}) \to 1$$

because the subgroup $\mathrm{Aut}(G, B, T, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\mathrm{Aut}(\widehat{G})$ maps bijectively onto $\mathrm{Out}(\widehat{G})$. Since we also have $\mathrm{Out}(\widehat{G}) \cong \mathrm{Aut}(X^\vee, R^\vee, R, \Delta^\vee) \cong \mathrm{Aut}(X, R, R^\vee, \Delta)$, we can form $^L G := \widehat{G} \rtimes \Gamma$.

For two connected reductive groups G_1 and G_2 their Langlands dual groups $^LG_1$ and $^LG_2$ are isomorphic as extensions of Γ if and only if G_1 and G_2 are inner forms of each other, and in this case they are even isomorphic as extensions endowed with conjugacy classes of distinguished splittings. The construction of the Langlands dual group is not functorial for arbitrary morphisms between connected reductive groups.

We can also define an analogue of parabolic subgroup of ^LG. It is such that a parabolic subgroup $P^0 \subseteq \widehat{G}$ such that its normalizer in ^LG maps onto Γ. For M Levi factor, we also have an embedding of extensions $^LM \to {}^LG$, see page 13 of the above note. It is well-defined up to conjugation by $\widehat{G}$, and in particular independent of the choice of parabolics P containing M. They are called the G-relevant Levis of ^LG.

We can now talk about Langlands parameters or Weil-Deligne Langlands parameters. There are three versions of it, see page 16 of the note. We denote by $\Phi(G)$ the $\widehat{G}$-conjugacy class of Langlands parameters.

For a semisimple elements, we can consider polar decomposition. This leads to the decomposition of a Langlands parameter and it's related to the Langlands classification. It is then natural to consider 'essentially discrete' parameters (^LG-irreducible parameters)

Relationship between Langlands parameters and Weil-Deligne Langlands parameters

Requirement on the Local Langlands correspondence:

Let $\Pi(G)$ be the set of isomorphism classes of admissible representations of $G(F)$ over $\mathbb{C}$. The tentative conjecture is that there should exist a map from $\Pi(G)$ to $\Phi(G)$ (Langlands parameter to be defined) satisfying the following property:

1. If G is a torus, then it should be the bijection deduced from class field theory.

2. The fiber should be finite (L-packets) and the image should contain all essentially discrete parameters.

3. Compatibility with product $G = G_1 \times G_2$

4. Compatibility with central isogeny ($\theta : G \to H$ induce map between L-group $\widehat{\theta} : {}^LH \to {}^LG$)

5. If $G = Res_{E/F}G_0$ then the LL-map for G and G_0 can be identified.

6. π is square-integrable iff $LL(\pi)$ is essentially discrete.

7. Compatibility with paraoblic induction

8. Compatibility with Langlands classification

9. Compatibility with Harish-Chandra character

They should imply compatibility with central character (we have a surjection $^L G \to {}^L Z$) and compatibility with twist by character $\chi : G(F) \to \mathbb{C}^\times$. They also tell us we can reduce to the discrete case. This also suggests the image of LL should be G-relevant parameters.

The unramified case: If G is unramified (extend to reductive group scheme over $\mathcal{O}_F$) and K is a hyperspecial compact open subgroup, then on K-unramified irreps of $G(F)$ the map LL is given by Satake isomorphism (See also here). On one hand, we know unramified representations are parametrized by rational Weyl orbits orbits under the rational Weyl group of continuous characters of $\chi : M_0(F) \to \mathbb{C}^\times$ where M_0 is an unramified torus (take the unique unramified constituent of $i_B^G \chi$) for any Borel subgroup B of G containing M_0. We then have $LL(\pi) = \iota_{M_0} \circ LL(\chi)$ by compatibility with parabolic induction when χ is unitary. In other words $LL(\pi)$ is the parameter associated to π by the Satake isomorphism. The hard part is to construct the explicit construction of an equivalence between $I(\chi)$ and $I(w\chi)$ where w is an element of the (relative) Weyl group.

Semisimplified correspondence: We should have one more property of the map LL (actually the semisimplified version $LL^{ss} : \pi \mapsto LL(\pi) \circ \iota_W$); It is essentially compatibility with supercuspidal support. The LLC evidently implies semisimplified LLC, but the reverse direction still seems open. What is missing is the fact that for any essentially square-integrable irreducible smooth representation π of $G(F)$, the semi-simplified parameter $LL^{ss}(\pi)$ comes from an essentially discrete Langlands parameter (which as above is automatically unique up to conjugation by the centralizer of $LL^{ss}(\pi)$ in $\widehat{G}$). The notion of essentially discrete Langlands parameter is purely algebraic (it does not rely on the topology of the coefficient field) so there ought to be a purely algebraic characterization of essentially squareintegrable representations. We check the validity of this intuition in the proposition 6.3 in Taibi's note.

We also see that from the requirement on LL^{ss} if $LL(\pi)$ is essentially discrete and trivial on SL_2, then π is supercuspidal. But the converse is not true, this is related to the the classification of essentially square-integrable representations in terms of supercuspidal representations (of Levi subgroups) is much more complicated in general than in the case of GL_n. In some applications having just the map LL is too crude, e.g. to formulate the global multiplicity formula for the automorphic spectrum of a connected reductive group over a global field, and so we would like to understand the fibers $\Pi_\phi(G)$. See Conjecture 6.4.

A Whittaker datum $\mathfrak{w}$ is (U, θ), where U is the unipotent radical of a Borel B and $\theta : U(F) \to \mathbb{C}^\times$ is a generic character. The adjoint group $G_{ad}(F)$ acts transitively on the set of such pairs. Shahidi's conjecture is that there should be a unique $\mathfrak{w}$-generic representation in every L-packet. The conjectural embedding $\iota_\mathfrak{w} : \Pi_\phi(G) \to Irr(\pi_0(\overline{S_\phi}))$ should map this unique generic representation to the trivial representation.

To characterize the embedding $\iota_\mathfrak{w}$ we introduce endoscopic data $\mathfrak{e}$. If $s \in S_\phi = Z(\widehat{G}, \phi)$ a semisimple element, we can construct $\Theta_{s,\phi}^\mathfrak{w}$, a a virtual character on $G(F)$ (take the weighted sum of the characters Θ_π for $\pi \in \Pi_\phi(G)$). For $s = 1$ we write it as $S\Theta_\phi$ and dropping $\mathfrak{w}$. We are going to perform the same trick as introduced by Lusztig, by introducing H, a quasi-split connected reductive group over F (dual to the connected component of the centralizer of s) an embedding $^L\eta : {}^L H \to {}^L G$ and a unique Langlands parameter $\phi_H : WD_F \to {}^L H$ for which ϕ factors. The virtual characters $\Theta_{\phi,s}^\mathfrak{w}$ and $S\Theta_{\phi_H}$ are related by the Langlands-Shelstad transfer factor $\Delta[\mathfrak{w}, \mathfrak{e}] : H(F)_{G-sr} \times G(F)_{sr} \to \mathbb{C}$. Using maximal tori and identifications of Weyl groups one can define a canonical map m from semisimple conjugacy classes in $H(F)$ to semisimple conjugacy classes in $G(F)$. An element $g \in G(\overline{F})$ is called strongly regular if its centralizer is a torus. A conjugacy class in $H(F)$ is called G-strongly regular if its image under m is strongly regular. There are a bunch of properties that m satisfies, including Γ-equivariance. Before we state the variance property that the transfer factor needs to satisfy, we need to understand $Irr(\pi_0(\widehat{T}^\Gamma))$ better. By Tate-Nakayama isomorphism, this is isomorphic to $H^1(F, T)$ via α_T functorial in T, and we can actually extend α_T to α_G where we replace the domain by $Irr(\pi_0(Z(\widehat{G})^\Gamma))$ for every connected reductive G (a family of maps of pointed sets which is a bijection in the case of non-Archimedean F). See Theorem 6.7 of Taibi's note. We can think of α_T as a pairing between an element of $H^1(F, T)$ and an element of $\pi_0(\widehat{T}^\Gamma)$. The variance property now becomes (6.3). For strongly regular element $\delta \in G(F)$, the set of $G(F)$-conjugacy classes $[\delta']$ which are stably conjugate to δ is parametrized by $ker(H^1(F, T) \to H^1(F, G))$ via $inv(\delta, \delta')$. It doesn't characterize it because it does not compare the values at unrelated matching pairs.

Now we have all the ingredients to characterize $\iota_\mathfrak{w}$. See conjecture 6.8. One of the most important condition is that the character $S\Theta_\phi : G_{sr}(F) \to \mathbb{C}$ should be invariant under stable conjugacy and independent of the Whittaker datum chosen.

Reference for representation of p-adic groups: Cartier, Renard, Zhu