Weil sheaves

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The etale $\overline{\mathbb{Q}_\ell}$-sheaves $\mathcal{F}_0$ on X_0 have a natural isomorphism $Fr^*\mathcal{F} \to \mathcal{F}$. The definition of Weil sheaves captures this feature, see page 7, section 1.1 def. 1.2 of Weil Conjectures, Perverse Sheaves & ℓ-adic Fourier Transform. The idea is that the Weil group acts on a Weil sheaf $\mathcal{F}$. More precisely, there is an equivalence between the category of lisse Weil sheaves and the category of continuous representations of the Weil group on finite dimensional vector spaces V over $\overline{\mathbb{Q}_\ell}$. The motivation is Galois descent theory tells us that there is an equivalence of categories between constructible $\overline{\mathbb{Q}_\ell}$-sheaves on X_0 and constructible $\overline{\mathbb{Q}_\ell}$-sheaves on X with a specified action of $Gal(X/X_0) = Gal(\overline{k}/k)$. But if we are in a setting where we know that such an isomorphism exists, but we cannot immediately verify that it is continuous, we are led to the definition of Weil sheaves.

The definition of Weil sheaves use geometric Frobenius, which induces a Frobenius automorphism (over $Spec(k)$ not $Spec(\overline{k})$), but the actions of Fr (relative Frobenius) and the Galois action of the Frobenius induced both on geometric points and on cohomology with compact support coincide so it doesn't matter.

An important theorem for Weil sheaves is that after twisting, they are essentially the same as etale $\overline{\mathbb{Q}_\ell}$-sheaves. First, by Proposition 1.1.12 of the note, a $\overline{\mathbb{Q}_\ell}$-representation (ρ, V) of $W(X_0, \overline{x})$ on $\mathcal{F}_{\overline{x}}$ extends to a representation of $\pi_1(X_0, \overline{x})$ iff some (equivalently, any) degree-1 element $f \in W$ acts with eigenvalues which are ℓ-adic units. We want to use this theorem to prove the following:

Suppose that X_0 is normal and geometrically connected. Then, an irreducible lisse Weil sheaf $\mathcal{F}_0$ of rank n is an actual $\overline{\mathbb{Q}_\ell}$-sheaf if and only if its determinant $\bigwedge^n \mathcal{F}_0$ is.

Note that the associated representation V of $W(X_0)$ is at least geometrically semisimple, i.e. its restriction to the geometric fundamental group $\pi_1(X)$ is semisimple since it is a continuous representation of a compact group. We can then apply Theorem 1.3.3 of the book to get that $G_{geom} :=$ Zariski closure of $\rho(\pi_1(X))$ is a semisimple algebraic group. The key theorem is Theorem 1.3.1, which says that the image of the geometric fundamental group under a continuous character of the Weil group is a finite group, whose proof uses geometric class field theory. See this note for the detail. This also implies for some $m > 0$, we can write $\sigma^m = g \cdot z$ for some $g \in G_{geom}$ and $z \in Z(G_{geom}\rho(W))$. Since G_{geom} is semisimple, by increasing m we can assume $\det(g) = 1$. Once we prove this, the rest boils down to proving all eigenvalues of g are ℓ-adic units. This is essentially a compactness argument, using the adjoint action of g on $\rho(\pi_1(X))$ (Note that g stablizes the image of the geometric fundamental group because σ and z does.)

Reference:

https://math.stanford.edu/~conrad/Weil2seminar/Notes/L19.pdf

https://math.stanford.edu/~conrad/Weil2seminar/Notes/L22.pdf