Perverse sheaves intro

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An excellent source of motivation is Morel's note. My handwritten note is here

Perverse sheaves is essentially an exotic t-structure on $D^b_c(X, R)$. The notion of t-exact functor is self-evident. The motivation for the study of one-sided t-exactness properties is that they lead to one-sided exactness properties on hearts. More precisely, if $\mathcal{A}_1$ is the heart of $\mathcal{D}_1$ and $F : \mathcal{D}_1 \to \mathcal{D}_2$ is a t-exact functor, then the fuctor obtained from composing inclusion and F and followed by taking H^0 is exact.

In defining perverse t-structure we use the notion of modified dimension $\mathrm{mdsupp}(\mathcal{F}) = \sup_{s \in \mathcal{T}} \dim X_s - \mathrm{grade}(\mathcal{F}|_{X_s})$ (See here for the notion of the grade of a module). This is needed when R is a general Noetherian commutative ring of finite global dimension, because $D^b(R\text{-mod}^{fg})^{\geq 0} = \{X \in D^b(R\text{-mod}^{fg}) : \mathrm{grade}(H^i(\mathbb{D}(X)) \geq i, \forall i\}$ (Prop. A. 10.6 of Achar's book). When R is a field, $\mathrm{mdsupp}(\mathcal{F}) = \dim \mathrm{supp}(\mathcal{F})$ and $\mathcal{F} \in {}^pD^b_c(X, R)^{\geq 0}$ iff $\mathbb{D}\mathcal{F} \in {}^pD^b_c(X, R)^{\leq 0}$.

When R is a field, the intersection cohomology construction produces the simple objects in the category of perverse sheaves. For general noetherian R, every perverse sheaf still admits a finite filtration by the intersection cohomology complexes. This fact is a step on the way to the proof of the fact that perverse sheaves form a noetherian abelian category.

The perverse t-structure has the following properties:

(Lemma 3.1.2) The categories ${}^pD^b_c(X, R)^{\leq 0}$ and ${}^pD^b_c(X, R)^{\geq 0}$ are closed under extension.

(Lemma 3.1.3) Let X be a smooth connected variety. Then the perverse t-structure are the usual t-structure shifted by $\dim(X)$ (This uses Lemma 2.8.2 describing the relationship between taking stalk and the duality functor $\mathbb{D}$.)

(Lemma 3.1.4) The following statements describe the interactions between perverse t-structure and open and closed embeddings $j : U \to X$ and $i : Z \to X$.

1. j^* and i_* are t-exact for the perverse t-structure.

2. $j_!$ and i^* are left t-exact.

3. j_* and $i^!$ are right t-exact (Note that $\mathbb{D} \circ j_* = j_! \circ \mathbb{D}$, this is a non-formal statement and requires a local calculation, see Lemma 2.8.5; it is used to prove the reflexiveness of $\mathbb{D}$, i.e. $\mathbb{D} \circ \mathbb{D} = id$ for any variety X. From this it follows that $\mathbb{D} \circ f_* = f_! \circ \mathbb{D}$ holds for any morphism $f : X \to Y$ of varieties and $\mathbb{D}$ also interwines f^* and $f^!$.)

(Lemma 3.1.6) This is the definition of perverse sheaves in Wikipedia in terms of stratifications (each stratum is smooth and on which $\mathcal{F}$ is a local system).

(Lemma 3.1.7) Let $\mathcal{F} \in D^b(X, R)$ be an object such that for each i, thesupport of $\mathcal{H}^i(\mathcal{F})$ is an algebraic variety, and that $\dim \mathrm{supp} \mathcal{H}^i(\mathcal{F}) \leq -i$. For all $\mathcal{G} \in {}^p D^b_c(X, R)^{\geq 1}$,we have $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ (proof uses Lemma 3.1.6 and induction on the number of strata).

(Theorem 3.1.9) The perverse t-structure is a bounded t-structure on $D^b_c(X, R)$.

(Prop. 3.1.10) The functor i_* induces an equivalence of categories $Perv(Z, R)$ with $\{\mathcal{F} \in Perv(X, R) : \mathrm{Supp}\mathcal{F} \subset Z\}$.

(Prop. 3.1.11 & 3.11.12) The functor $\mathbb{D}$ and f_* for finite morphism $f : X \to Y$ are t-exact for the perverse t-structure.

Intersection cohomology complexes: For locally closed embedding $h : Y \to X$ we have the following functor that "interpolates" between $h_!$ and h_*:
$h_{!*} : Perv(Y, R) \to Perv(X, R)$ given by
$h_{!*}(\mathcal{F}) := im({}^p H^0(h_! \mathcal{F}) \to {}^p H^0(h_* \mathcal{F}))$ We emphasize that the following functor is not defined for arbitrary objects in $D^b_c(Y, R)$.

The key results about intersection cohomology complexes is the following (Exercise 3.1.6):

Let $h : Y \to X$ be a locally closed embedding. Let $\mathcal{F}$ be a perverse sheaf on Y, and let $\mathcal{G}$ be a perverse sheaf on X that is supported on $\overline{Y} \setminus Y$. Show that
$\mathrm{Hom}({}^p H^0(h_! \mathcal{F}), \mathcal{G}) = \mathrm{Ext}^1({}^p H^0(h_! \mathcal{F}), \mathcal{G}) = 0$ and that
$\mathrm{Hom}(\mathcal{G}, {}^p H^0(h_* \mathcal{F})) = \mathrm{Ext}^1(\mathcal{G}, {}^p H^0(h_* \mathcal{F})) = 0.$

This is not true for the ordinary t-structure.

(Lemma 3.3.2) Let $h : Y \to X$ a locally closed embedding. we have

1. For $\mathcal{F} \in Perv(Y, k)$, there is a natural isomorphism $h^* h_{!*} \mathcal{F} = \mathcal{F}$.
2. The object $h_{!*} \mathcal{F}$ has no nonzero subobjects or quotient objects supported on $\overline{Y} \setminus Y$.

(Lemma 3.3.3) Characterization of $h_{!*} \mathcal{F}$ by Lemma 3.3.2 (2).

(Lemma 3.3.7 & 3.3.8) Let X be an irreducible variety. Let $j : U \to X$ be the inclusion map of an open subset, and let $i : Z \to X$ be the complementary closed subset. Let $\mathcal{F}$ be a perverse sheaf on X.

1. If $\mathcal{F}$ has no quotient supported on Z, then there is a natural short exact sequence $0 \to {}^pH^0(i_* i^! \mathcal{F}) \to \mathcal{F} \to j_{!*}(\mathcal{F}|_U) \to 0$.
2. If $\mathcal{F}$ has no subobject supported on Z, then there is a natural short exact sequence $0 \to j_{!*}(\mathcal{F}|U) \to \mathcal{F}^p H^0(i_* i^* \mathcal{F}) \to 0$.

This is the analogue of the open-closed SES for ordinary sheaves.

Definition 3.3.9: Intersection complexes $IC(Y, \mathcal{L}) := h_{!*}(\mathcal{L}[\dim Y])$ for $Y \subset X$ smooth, locally closed subvariety and $\mathcal{L}$ local system of finite type. Special case when $Y = X_{sm}$ and $\mathcal{L} := \underline{k}$ denoted by $IH^k(X, k)$.

Lemma 3.3.11: Criterion for when $\mathcal{F} \cong IC(X_u, \mathcal{L})$ for a stratification w.r.t. $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible.

Lemma 3.3.12: For $U \subset X$ open and X smooth and $\mathcal{L}$ a local system on X, we have $IC(U, \mathcal{L}|_U) \cong \mathcal{L}[n]$.

The proof uses Lemma 3.3.11, as in the proof of Lemma 3.3.13 and Lemma 3.3.14 (dual and tensor product of IC).

Noetherian property for perverse sheaves:

(Prop. 3.4.1) The category $Loc^{ft}(X, k)[n]$ for a smooth connected subvariety of dimension n is a Serre subcategory of $Perv(X, k)$.

The key is to show this is closed under taking subobjects. The idea is to use Lemma 3.3.12.

(Theorem 3.4.2) Every perverse sheaf admits a finite filtration whose subquotients are IC complexes.

The idea is keep using Lemma 3.3.7 and 3.3.8 to break things down into IC complexes.

(Theorem 3.4.4) The category $Perv(X, k)$ is Noetherian.

(Theorem 3.4.5) Assume k is a field, then the category $Perv(X, k)$ is also Artinian and the simple objects are $IC(Y, \mathcal{L})$ where $\mathcal{L}$ is an irreducible local system and Y is a smooth locally closed subvariety.

Theorem: If $f : X \to Y$ is a smooth morphism of relative dimension d. The functor $f^*[d] \cong f^![-d](-d) : D^b_c(Y, R) \to D^b_c(X, R)$ is t-exact for the perverse t-structure.

Introduce $f^\dagger := f^*[d]$ and $f_\dagger := f_*[-d]$. We have the following important theorem:

(Thm. 3.6.6) For smooth surjective morphism f, the functor $f^\dagger$ is faithful and if f has connected fiber, then $f^\dagger$ is fully faithful.

For shifted local system this can be understood in terms of restriction of representation of π_1. See Remark 3.6.7.

(Corollary 3.6.9) If k is a field, then $f^\dagger$ sends any simple perverse sheaf to simple perverse sheaf.

Section 3.7: Perverse sheaves admit smooth descent.

General result on constructible sheaves:

It is easy to see that f^* and $\otimes^L$ preserve constructibility. For f_* and $f_!$, consult Theorem 2.7.1 of Achar's book. The idea is to use Nagata's compactification theorem to reduce to the open embedding case and the proper case. In the proper case we need to use some structure theorem like resolution of singularities. For $R\mathcal{H}om$, see Proposition 2.7.3. Finally for $f^!$ it is a consequence of $\mathbb{D} \circ \mathbb{D} = id$ and $f^! \circ \mathbb{D} = \mathbb{D} \circ f^*$.

(Cohomological vanishing result) Let X be a variety of dimension n, and $\mathcal{F}$ be a constructible sheaf on X. The R-modules $H_c^k(X, \mathcal{F})$ and $H^k(X, \mathcal{F})$ are finitely generated for all k, and it vanishes unless $0 \leq k \leq 2n$.

Reference: Achar's book, Goresky's note (excellent for the motivation of perverse sheaves and the sheaf-theoretic definition)