Weil I, Lefschetz pencil/fibration, Picard-Lefschetz, Main lemma

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Note for proof of functional eq. and rationality

See here for the reduction of Riemann hypothesis to the case of middle cohomology and even dimension by tensor power trick.

The main geometric input for Weil I is Lefschetz pencil and Picard-Lefschetz formula, which is a complex analogue of Morse theory. The idea is we induct on the dimension on the variety $\dim X_0 = n+1$, and we can assume there is a Lefscetz fibration $\pi : X \to \mathbb{P}^1$ (more precisely we get a fibration $X \setminus B \to \mathbb{P}^1$ where B is the base locus of the pencil; but we can extend this map to the blow up $\tilde{X} := Bl_B X$ and there is an injection $H^*(X) \to H^*(\tilde{X})$), so by the Leray spectral sequence it is enough to understand eigenvalues of Frobenius on $H^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell)$, $H^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$ and $H^0(\mathbb{P}^1, R^{n+1}\pi_*\mathbb{Q}_\ell)$.

For H^2, note that $R^r\pi_*\mathbb{Q}_\ell$ is locally constant on $\mathbb{P}^1$ if $r \neq n, n+1$. This essentially says that the vanishing cycles only show up in middle cohomology degree. Topologically we can see that the preimage of the node under the deformation retract onto the singular fiber (called the Milnor fiber) is a bouquet of spheres. Hence $H^2(\mathbb{P}^1, R^{n-1}\pi_*) = H^{n-1}(\text{fiber})(-1)$. To use the induction hypothesis we can further slice the fiber by a hyperplane section Z and use the weak Lefschetz theorem that $H^{n-1}(\text{fiber})$ injects into $H^{n-1}(Z)$. For H^0, it divides into two cases. If the vanishing cycles are nonzero, then $R^{n+1}\pi_*$ will be locally constant as well, and we can use the dual weak Lefschetz. If it is zero, the picture is a family of double tori degenerate into two (genus 1) tori, see the picture in Litt's note. In this case we have a SES that also allows us to use the induction hypothesis, see part B of proof of Theorem 7.1 (page 25) in Milne's translation of Weil I.

The existence of Lefschetz pencil is guaranteed by Bertini's theorem, an algebro-geometric version of the Morse lemma. See also Milne's note for a proof.

The next important ingredient is the Picard-Lefschetz formula computing the monodromy around the critical point of the fibration. The action is essentially by Dehn twist, see this slide by Jonathan Evans for a nice illustration.

Definition of the vanishing cycle (well-defined up to sign): See SGA 7.2 XIV, section 3, page 146-157. The idea is that the vanishing cycle should be a generator of the middle cohomology of the Milnor fiber.

Note: For the Legendre family of elliptic curve $y^2 = x(x-s)(x-1)$. Note that $(x, y, s) \mapsto s$ is not differentiable at $s = 0$ (since it looks like $\sqrt{x^2 + y^2}$ near $s = 0$, but the family $y^2 = (x^2 - s)(x-1)$ is, for which we can apply Picard-Lefschetz formula to compute the monodromy. Since when s circles around 0 once s^2 circles around 0 twice, so that's why the local monodromy at $s = 0$ of the Lengendre family is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2$. For the more analytic point of view see Singularities of Differentiable Maps, Volume 1: Classification of Critical Points section 1.3. See also the book Period mappings and period domains page 19 for visualization.)

We would like to understand better the sheaf $R^n \pi_* \mathbb{Q}_\ell$. The fact that the monodromy acts abelianly is an indication of the tameness of the ramification of $R^n \pi_* \mathbb{Q}_\ell$ at $s \in S$. From Picard-Lefschetz we see the space $E \subseteq H^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ spanned by vanishing cycles is stable under $\pi_1(L \setminus S)$ and $E^\perp$ is the invariant of the monodromy. The space H^n admits a filtration $0 \subseteq E \cap E^\perp \subseteq E \subseteq H^n$ and the representation $E \cap E^\perp$ and H^n/E are constant, so the only interesting part is $E/E \cap E^\perp$.

This representation has big monodromy and hence satisfies the assumption of the main lemma. For proof of Main lemma, see Litt's note. The idea is that the zeta function of a locally constant sheaf on a curve takes a particular easy form after applying the trace formula, with the radius of convergence controlled by H_c^2, which turns out to be certain coinvariant that is insensitive to passing to Zariski closure. Thus if the corresponding representation E of π_1 is Zariski dense in Sp, then we are reduced to computing the coinvariant of symplectic group, which can be found in Weyl's The Classical Groups.