Topological K-theory

J'ignore • 5 Dec 2025

Homotopy invariance of vector bundles: If X is compact Hausdorff and $f \simeq g$, then $f^*E \cong g^*E$. Let H be a homotopy between f and g, then $\pi^*H(-,t)^*(E)$ ($\pi : X \times I \to X$ is the projection map) and $H^*(E)$ on $X \times \{t\}$ are isomorphic. Let s be such an isomorphism. Then s is a section of $Hom(\pi^*H(-,t)^*E, H^*(E))$ on $X \times \{t\}$, so we can extend over all of $X \times I$. Note that in an open neighborhood of $X \times \{t\}$, we still get isomorphism. By compactness of X, it includes $X \times (a_t, b_t)$, so the isomorphism type of $H^*(E)|_{X \times \{t\}}$ is locally constant in t, hence over all of I. It remains true for paracompact base. The functor sending X to rank k-vector bundles on X is representable by $Gr_k(\mathbb{C}^\infty)$. The key point is that $Isom(\mathbb{C}^\infty \oplus \mathbb{C}^\infty, \mathbb{C}^\infty)$ is contractible (Eilenberg-Mazur Swindle). The $\{Gr_k(\mathbb{C}^\infty)\}_{k \in \mathbb{N}}$ forms an algebra over the operad $Isom((\mathbb{C}^\infty)^{\oplus k}, \mathbb{C}^\infty)$.

If we work over a compact base X, then we just need to invert the trivial bundle to form $K^0(X)$ since $X \to Gr_k(\mathbb{C}^\infty)$ factors through some $Gr_k(\mathbb{C}^N)$ and we can take the orthogonal complement bundle E' of E which satisfies $E \oplus E' \cong X \times \mathbb{C}^N$. If we further mod out by the trivial bundle, then this reduced K-group $\tilde{K}^0(X)$ (vector bundlues modulo stable equivalence) is isomorphic to $[X, BU]$ where $BU := \varinjlim_k Gr_k(\mathbb{C}^\infty)$. We can start build a cohomology theory $(X, A) \mapsto \tilde{K}^0(X/A)$ and define $K^{-n}(X) := \tilde{K}^0(\Sigma^n X)$. Note that we can rewrite $BU(k) = \varinjlim_N U(N)/(U(k) \times U(N-k)) \simeq (\varinjlim_N U(N)/U(N-k))/U(k)$. The upshot is that $\varinjlim_N U(N)/U(N-k)$ is contractible by computing its hommotopy group, since $U(N-k) \to U(N)$ induces an isomorphism on π_* for $* < 2(N-k)$ (by the fiber sequence), hence every homotopy group is zero when pass to the direct limit.

We can use Borel's transgression theorem to compute the cohomology of $BU(n)$ (which is simply connected since its fundamental group is the component group of unitary group, which is trivial). Since $H^*(U(n))$ is an exterior algebra on $x_1, x_3, ...x_{2n-1}$ and they are transgressive by induction on n, we get $H^*(BU(n)) = \mathbb{Z}[c_1, ...c_n]$ where c_i is the transgression of x_{2i-1} and $\deg(c_i) = 2i$.

Bott periodicity: $B(\mathbb{Z} \times BU) \simeq U$. For a short proof see here. See here for discussion of Bott periodicity. Since $\Omega(BG) \simeq G$ (see here), so $\mathbb{Z} \times BU \simeq \Omega U$ and $\Omega^2(\mathbb{Z} \times BU) \simeq \Omega^2(BU) \simeq \Omega(U) \simeq \mathbb{Z} \times BU$. Using the double periodicity, we can define $K^1(X) := [X, U]$ and it constitutes a generalized cohomology theory.

Characteristic class: https://en.wikipedia.org/wiki/Characteristic_class (supposed to be a contravariant theory detecting existence of section)

Reference:

https://people.math.harvard.edu/~dafr/M392C-2012/Notes/lecture6.pdf