More on Frobenii

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We list some recurring remarks about Frobenii:

1. The induced map of the absolute Frobenius on cohomology is the identity, because the induced map on etale site $Frob_{abs} : X_{et} \to X_{et}$ is naturally isomorphic to id via the relative Frobenius. This also shows that for the relative Frobenius, the induced map on etale site (hence cohomology) is invertible (this also follows from it being a universal homeomorphism) whose inverse is the induced map from the arithmetic Frobenius.

2. As a corollary of the previous point, we also see that the geometric Frobenius and the relative Frobenius induce the same map on etale site (hence etale cohomology).

3. We need to use the geometric Frobenius/relative Frobenius in the Lefschetz fixed point formula, since LHS doesn't even make sense for $Frob_{abs}$ which is not a $\overline{k}$-morphism.

4. For U etale over X, the relative Frobenius $F_{X/U} : U \to Frob_{abs}^{-1}(U)$ is an isomorphism (because it is a radical surjection between two etale X-scheme; for detail see here). It is not if U is not etale, e.g. take $U := X$ and $X := Spec(\overline{\mathbb{F}_q})$. This also explains why if $\mathcal{F}_0$ is an etale sheaf over X_0, then there is an isomorphism $Fr_{\mathcal{F}_0} : F_{X_0}^* \mathcal{F}_0 \to \mathcal{F}_0$ (see Proposition 4 of the above note; it is the adjoint of the isomorphism $F_{X/U}$ where $U := [\mathcal{F}]$), so we have an (invertible) endomorphism of cohomology $F : H^i(X, \mathcal{F}) \xrightarrow[(F_{X_0} \times id)^*]{} H^i(X, (F_{X_0} \times id)^* \mathcal{F}) \xrightarrow[Fr_{\mathcal{F}_0}]{} H^i(X, \mathcal{F})$. This coincides with the action of geometric Frobenius on H^i induced by $(Id_{X_0} \times Frob_{\overline{\mathbb{F}_q}}^{-1})$.

5. We need generalization of the Lefschetz fixed point to (i) non-proper schemes (ii) non-constant sheaves (iii) $\mathbb{Z}/\ell^n$-modules. The relative version can be expressed in terms of L-function $L(X, \mathcal{F}, t)$ (defined by multiplying the local factors): see Theorem 8 of the above note.

6. See Theorem 12 of the above note for Deligne's purity theorem (without assumptions on smoothness and properness; it essentially says that $R^n f_!$ increases weight by at most n for a separated map $f : X \to S$ of finite type k-schemes) and how to derive Riemann hypothesis from it (essentially use Poincare duality to show purity).

Reference:

https://mathoverflow.net/questions/30302/geometric-vs-arithmetic-frobenius (See the comment below the answer for the importance of the universal nature of the absolute Frobenius.)

https://www.math.mcgill.ca/goren/SeminarOnCohomology/Frobenius.pdf

https://stacks.math.columbia.edu/tag/03SL (reference for absolute frobenius induce identity on etale cohomology)