Disjointness theorem

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See here for the property of etale cohomology of quotient variety. The key point is that when we reduce to $\mathbb{Z}/\ell^n\mathbb{Z}$, there may be cohomology groups, but they are all killed by $|G|$, and hence stay so after passing to the limit over n, so that when inverting ℓ, they disappear.

We would like to show geometric conjugacy class of (T, θ) (T is a F-stable maximal torus and θ is a character of T^F) gives a partition of $Irr(G^F)$, where we say (T, θ) and (T', θ') are geometrically conjugate if there exists some n such that $(T, \theta \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ and $(T', \theta' \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ are G^{F^n}-conjugate. The problem is that even $\langle R_T^\theta, R_{T'}^{\theta'}\rangle = 0$ they could still share irreducible characters since they are virtual characters.

Theorem (DM 13.3) If χ is a common irreducible constituent of $H_c^i(\mathcal{L}^{-1}(U)) \otimes_{\overline{\mathbb{Q}_\ell}[T^F]} \theta$ and $H_c^j(\mathcal{L}^{-1}(U')) \otimes_{\overline{\mathbb{Q}_\ell}[T^F]} \theta'$, then the pairs (T, θ) and (T', θ') are geometrically conjugate.

Proof: Let $\overline{\theta}$ be the conjugate representation (isomorphic to the dual representation) and $\theta^\vee$ be the right T^F-representation obtained from θ. The assumption implies that $\overline{\theta} \otimes \theta'^\vee$ occurs in $H_c^i(\mathcal{L}^{-1}(U)^\vee) \otimes_{G^F} H_c^j(\mathcal{L}^{-1}(U'))$. Consider the variety $\mathcal{L}^{-1}(U) \times_{G^F} \mathcal{L}^{-1}(U')$ where the notation means we quotient by the action $g \cdot (x, x') = (gx, gx')$. By Kunneth we know that the tensor product of cohomology is isomorphic to the cohomology of this variety as a T^F-module-T'^F.

To understand the cohomology of this variety, we first note that this is isomorphic to $Z = \{(u, u', g) \in U \times U' \times G : uF(g) = gu'\}$, where the action is mapped to $(t, t') \cdot (u, u', g) = (tut^{-1}, t'^{-1}u't', tgt')$. The isomorphism is given by $(x, x') \mapsto (\mathcal{L}(x), \mathcal{L}(x'), x^{-1}x')$.

We further decompose the variety Z into Z_w where w runs through the set of intertwiner from T to T' (more precisely $z \in T \setminus \mathcal{S}(T, T')/T'$ where $\mathcal{S}(T, T') := \{w \in G : T = wT'w^{-1}\}$). For this we use the Bruhat decomposition $G = \bigsqcup_{w \in T\setminus\mathcal{S}(T,T')/T'} F^{-1}(B)wF^{-1}(B')$.

Next we replace U' and B' by its twist by w. More precisely, we introduce the variables $w' = wF(w)^{-1}, F' = w'F$, then

$$Z_w \cong Z'_w \ := \ \Big\{(u_1, u'_1, g_1) \in U \times wU'w^{-1} \times F^{-1}(B)\,(F')^{-1}(wB'w^{-1}) : \ u_1 F(g_1$$

The actoion of $T^F \times (T')^F$ now transfers to an action of $T^F \times (wT'w^{-1})^{F'}$. The benefit of doing this is this resembles more the transversal appearing in Mackey Formula. Now we can forget w and redefine $B' := wB'w^{-1}$, $T' := wT'w^{-1}$ and $U' := wU'w^{-1}$. We are going to use the decomposition $F^{-1}(B)(F')^{-1}(B') = F^{-1}UTT'(F')^{-1}(U')$.

Introduce

$$Z''_w := \Big\{(u, u', u_1, u'_1, n) \in U \times U' \times F^{-1}(U) \times (F')^{-1}(U') \times TT' \ \Big| \ uF(n) = u_1 n u'_1$$

There is an affine fibration $\pi : Z''_w \to Z'_w$ given by $(u, u', u_1, u'_1, n) \mapsto (uF(u_1)^{-1}, u'F'(u'_1), u_1 n u'_1)$, so the cohomology of Z''_w is isomorphic to that of Z'_w as a T^F-module-$T'^{F'}$ if we equip Z''_w with the action $(t, t') \cdot (u, u', u_1, u'_1, n) \mapsto (tut^{-1}, t'^{-1}u't', tu_1t^{-1}, t'^{-1}u'_1t', tnt')$.

The idea now is to find a big enough torus H_w° commuting with the action of $T^F \times (T')^{F'}$, since we know the vitural representation $H^*(Z''_w)$ and $H^*(Z''^{H_w^\circ}_w)$ are isomorphic. A natural idea is to take $T = Z(T)$ and $T' = Z(T')$, but for the action of (t, t') to take Z''_w to itself, it is easy to check that this happens iff $F(n)^{-1}(l^{-1}F(l))F(n) = w'^{-1}(mF'(m^{-1}))w'$. If we write $n = \lambda\mu$, then since T, T' are commutative and T (resp. T') is F-stable (resp. F'-stable), we can check this is the same as $l^{-1}F(l) = w'^{-1}(mF'(m^{-1}))w'$.

We compute that $H_w^\circ = \{(N_{F^n/F}(\tau), N_{F'^n/F'}(w'\tau^{-1}w')) : \tau \in T\}$, where n is such that $F^n(w) = w$ (and for this n we also have $F'^n = F^n$ and $F^n(w') = w'$).

https://www.math.columbia.edu/~chaoli/docs/LusztigClassification.html#ref-1