Disjointness theorem

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See here for the property of etale cohomology of quotient variety. The key point is that when we reduce to $\mathbb{Z}/\ell^n\mathbb{Z}$, there may be cohomology groups, but they are all killed by $|G|$, and hence stay so after passing to the limit over n, so that when inverting ℓ, they disappear.

We would like to show geometric conjugacy class of (T,θ) (T is a F-stable maximal torus and θ is a character of T^F) gives a partition of $Irr(G^F)$, where we say (T,θ) and (T',θ') are geometrically conjugate if there exists some n such that $(T,\theta\circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ and $(T',\theta'\circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ are G^{F^n}-conjugate. The problem is that even $\langle R_T^\theta, R_{T'}^{\theta'}\rangle = 0$ they could still share irreducible characters since they are virtual characters.

Theorem (DM 13.3) If χ is a common irreducible constituent of $H_c^i(\mathcal{L}^{-1}(U))\otimes_{\overline{\mathbb{Q}_\ell}[T^F]}\theta$ and $H_c^j(\mathcal{L}^{-1}(U'))\otimes_{\overline{\mathbb{Q}_\ell}[T^F]}\theta'$, then the pairs (T,θ) and (T',θ') are geometrically conjugate.

Proof: Let $\overline{\theta}$ be the conjugate representation (isomorphic to the dual representation) and $\theta^\vee$ be the right T^F-representation obtained from θ. The assumption implies that $\overline{\theta}\otimes\theta'^\vee$ occurs in $H_c^i(\mathcal{L}^{-1}(U)^\vee)\otimes_{G^F}H_c^j(\mathcal{L}^{-1}(U'))$. Consider the variety $\mathcal{L}^{-1}(U)\times_{G^F}\mathcal{L}^{-1}(U')$ where the notation means we quotient by the action $g\cdot(x,x')=(gx,gx')$. By Kunneth we know that the tensor product of cohomology is isomorphic to the cohomology of this variety as a T^F-module-T'^F.

To understand the cohomology of this variety, we first note that this is isomorphic to $Z=\{(u,u',g)\in U\times U'\times G : uF(g)=gu'\}$, where the action is mapped to $(t,t')\cdot(u,u',g)=(tut^{-1},t'^{-1}u't',tgt')$. The isomorphism is given by $(x,x')\mapsto(\mathcal{L}(x),\mathcal{L}(x'),x^{-1}x')$.

We further decompose the variety Z into Z_w where w runs through the set of intertwiner from T to T' (more precisely $z\in T\setminus\mathcal{S}(T,T')/T'$ where $\mathcal{S}(T,T'):=\{w\in G : T=wT'w^{-1}\}$). For this we use the Bruhat decomposition $G=\bigsqcup_{w\in T\setminus\mathcal{S}(T,T')/T'}F^{-1}(B)wF^{-1}(B')$.

Next we replace U' and B' by its twist by w. More precisely, we introduce the variables $w'=wF(w)^{-1}$, $F'=w'F$, then

$$Z_w \cong Z'_w := \left\{(u_1, u'_1, g_1) \in U \times wU'w^{-1} \times F^{-1}(B)\,(F')^{-1}(wB'w^{-1}) : \; u_1 F(g_1)\right.$$

The actoion of $T^F \times (T')^F$ now transfers to an action of $T^F \times (wT'w^{-1})^{F'}$. The benefit of doing this is this resembles more the transversal appearing in Mackey Formula. Now we can forget w and redefine $B' := wB'w^{-1}$, $T' := wT'w^{-1}$ and $U' := wU'w^{-1}$. We are going to use the decomposition
$F^{-1}(B)(F')^{-1}(B') = F^{-1}UTT'(F')^{-1}(U')$.

Introduce

$$Z''_w := \left\{(u, u', u_1, u'_1, n) \in U \times U' \times F^{-1}(U) \times (F')^{-1}(U') \times TT' \;\middle|\; uF(n) = u_1 n u'_1\right.$$

There is an affine fibration $\pi : Z''_w \to Z'_w$ given by
$(u, u', u_1, u'_1, n) \mapsto (uF(u_1)^{-1}, u'F'(u'_1), u_1 n u'_1)$, so the cohomology of Z''_w is isomorphic to that of Z'_w as a T^F-module-$T'^{F'}$ if we equip Z''_w with the action $(t, t') \cdot (u, u', u_1, u'_1, n) \mapsto (tut^{-1}, t'^{-1}u't', tu_1t^{-1}, t'^{-1}u'_1t', tnt')$.

The idea now is to find a big enough torus H°_w commuting with the action of $T^F \times (T')^{F'}$, since we know the vitural representation $H^*(Z''_w)$ and $H^*(Z''^{H^\circ_w}_w)$ are isomorphic. A natural idea is to take $T = Z(T)$ and $T' = Z(T')$, but for the action of (t, t') to take Z''_w to itself, it is easy to check that this happens iff $F(n)^{-1}(l^{-1}F(l))F(n) = w'^{-1}(mF'(m^{-1}))w'$. If we write $n = \lambda\mu$, then since T, T' are commutative and T (resp. T') is F-stable (resp. F'-stable), we can check this is the same as $l^{-1}F(l) = w'^{-1}(mF'(m^{-1}))w'$.

We compute that $H^\circ_w = \{(N_{F^n/F}(\tau), N_{F'^n/F'}(w'\tau^{-1}w')) : \tau \in T\}$, where n is such that $F^n(w) = w$ (and for this n we also have $F'^n = F^n$ and $F^n(w') = w'$).

https://www.math.columbia.edu/~chaoli/docs/LusztigClassification.html#ref-1