Kunneth formula, cycle class map, Lefschetz trace formula

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Reference: Milne's note section 22, 23 and Litt's note

The upshot is that Kunneth formula is a consequence of projection formula and proper base change (or smooth base change if the prime ℓ in question is invertible on the base)

The cycle class map $C^r(X) \to H^{2r}(X, \Lambda(r))$ for $r = 1$ is just the connecting homomorphism associated to the Kummer sequence. The key input is purity, which says that $H^0(Z, \Lambda) \xrightarrow{\cong} H^{2r}_Z(X, \Lambda(r))$ for Z smooth (the proof hinges on the smooth pair $(\mathbb{A}^m, \mathbb{A}^{m-c})$). For Z singular we can still define it thanks to the isomorphism $H^{2c}_Z(X; \Lambda) \cong H^{2c}_{Z \setminus Y}(X \setminus Y; \Lambda)$ where $Y = Z_{sing}$.

Construction of Chern class: The idea is to introduce the projectivization $\mathbb{P}(E)$ of a vector bundle $E \to X$, which allows us to reduce to the case of line bundles by Grothendieck's axioms. For detail see the Wikipedia section. See also [here](https://en.wikipedia.org/wiki/Chern_class#In_algebraic_geometry for the algebrogeometric analogue of the cohomology ring, the Chow ring. See also Milne's note, especially page 141 for how the cycle class map is related to Chern class and chern character (we define the Chern character with denominators because $x_r \cdot x_s := \frac{(r+s-1)!}{(r-1)!(s-1)!}$ should be the correct multiplicative structure on $H^*(X)$ to make $grK^* \to H^*(X)$ into a ring homomorphism).

Digression to Grothendieck-Riemann-Roch: Essentially GRR is about the failure for the Chern character to commute with pushforward. It involves the Todd class as a correction factor, see the intuition here and also how you can discover the Todd class.

Lefschetz trace formula:

Example: $Tr(\varphi^*|H^2(\mathbb{P}^1_{\mathbb{F}_q}, \mathbb{Q}_\ell)) = n$ where $\varphi : x \mapsto x^n$ is the power-n map. To see this, note that $H^2(\mathbb{P}^1_{\mathbb{F}_q}, \mathbb{Z}/\ell^m)$ is cokernel of $Pic \xrightarrow{\times \ell^m} Pic$, so we need to understand the action of φ^* on line bundles on $\mathbb{P}^1$. But a line bundle on $\mathbb{P}^1$ is given by transition functions x^k, and φ^* essentially plug in x^n in place of x.

Reference: https://math.stanford.edu/~conrad/Weil2seminar/Notes/L20.pdf