Artin comparison theorem, etale fundamental group, l-adic cohomology

J'ignore • 23 Nov 2025

We always have $H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(\pi_1^{et}(X),\mathbb{Z})$ (here by $\pi_1^{et}(X)$ we mean the big etale group). See the discussion here, especially Emerton's comment that even etale cohomology with $\mathbb{Z}$ coefficients computes the wrong thing in degrees greater than 0, it turns out although $H^1(X,\mathbb{Z})$ is in cohomological degree 1, it is in motivic weight 0, and this is why etale cohomology can detect it. See also this mathoverflow answer.

To define ℓ-adic cohomology, we cannot use the coeffcient sheaves $\underline{\mathbb{Z}_\ell}$ directly, essentially due to the vanishing of the continuous Galois cohomology $H^i_{cont}(G_k,F)$ for any torsion-free constant abelian sheaf F. For the detail see this note. Note that the normality on X is crucial in ensuring $F \to \eta_*\eta^*F = \eta_*F$ to be an isomorphism for the generic point η of X.

When $\mathcal{X}$ is normal, flat and proper over a DVR R, the map $\pi_1^{et}(\mathcal{X}_{\overline{K}},\overline{\eta}) \to \pi_1^{et}(\mathcal{X}_{\overline{k}},\overline{\xi})$ is surjective. The content of this is that given $\mathcal{Y} \to \mathcal{X}$ finite etale with $\mathcal{Y}$ connected, then $\mathcal{Y}_K$ is also connected. For more about fundamental group of normal scheme, see here. The normality assumption in particular ensures that if two etale X-schemes E and F are generically isomorphic, then they are isomorphic, since both can be seen as normaliztion of X in the function field of their generic fibers.

Artin comparison theorem doesn't state that the category of sheaves $Shv(X_{et})$ is equivalent to $Shv(X_{et,an})$ for a connected finite type scheme X over $\mathbb{C}$. It only states that for $\mathcal{F}$ constructible, the i-th cohomology of $\mathcal{F}$ computed in the two sites are isomorphic for all i. The idea behind the proof is the notion of elementary fibrations, which essentially allows us to reduce to the case of $i = 0$ and $i = 1$. For $i = 1$ this is Riemann's existence theorem. See Milne's note, section 21 for more details. One can also use Daniel Litt's note, though I think the proof is slightly insufficient in that it's not clear why $R^if_*\mathcal{F}$ is constructible since f is not proper (but maybe it's my own stupidity).

Note that the statwment is not true if $\mathcal{F} = \mathbb{Z}$. However, it is still true if X is not normal, see SGA1, chapter 5, theorem 5.1.

Note that we have a map from the group cohomology $H^i(\pi_1, M)$ to the etale cohomology $H^i(X_{et}, M)$, but that's only an isomorphism for $i = 0, 1$ (similar to what happens in the topological setting). See this note. For $M = \underline{\mathbb{Z}/p}$ and π_1^{et} acts trivially, this is the correspondence between $\mathbb{Z}/p$-torsors and $\mathrm{Hom}(\pi_1^{et}, \mathbb{Z}/p)$. There is another proof in Alex Youcis's note, Theorem 6. The key is Lemma 5, which states that extension of lcc sheaves by lcc sheaves are lcc.

Now, one test of the usefulness of such a theory will be the existence of theorems for these sheaves analogous to those we had for LCC sheaves. In particular, we're going to want an equivalence of categories between 'locally constant $\mathbb{Z}_\ell$-sheaves' and the category continuous representations $\pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{GL}_n(\mathbb{Z}_\ell)$.

Note that an lcc $\mathbb{Z}_\ell$-sheaf (i.e. each sheaf in the projective system is lcc) not locally constant in the sense that there is a cover triviallizing it since the cover can get finer and finer. To define lisse $\mathbb{Q}_\ell$-sheaves, we mimic the definition of isogeny category, and we will get an equivalence of categories between lisse $\mathbb{Q}_\ell$-sheaves and continuous representations of π_1^{et} valued in $GL_n(\mathbb{Q}_\ell)$ (the content is that every such representation is conjugate to one valued in $GL_n(\mathbb{Z}_\ell)$). See the above note for more detail.

Reference:

https://math.stanford.edu/~conrad/Weil2seminar/Notes/L15.pdf

https://math.stanford.edu/~conrad/Weil2seminar/Notes/L16.pdf

https://math.stanford.edu/~conrad/Weil2seminar/Notes/L17.pdf