Lefschetz hyperplane theorem, characteristic class, Steenrod square

J'ignore • 14 Nov 2025

The intuition of Lefschetz hyperplane theorem comes from its proof using Morse theory, in particular Andreotti–Frankel theorem, saying that an smooth, complex affine variety of complex dimension n has homotopy type of a (real) n-dimensional CW complex (This implies vanishing of cohomology for affine complex variety above n.). The statement then follows by applying to $Y = X \setminus D$ and Alexander-Lefschetz duality. The idea is to use excision applied to a ANR (to be able to use excision) together with Poincare duality. See Hatcher, theorem 3.44. See the derivation here.

Cohomology with compact support: $H^i_c(A) = \varinjlim_K H^i(A, A \setminus K)$. They are the set of cochains which vanishes outside a compact set K.

Relationship with Gysin sequence: The idea is that Thom space gives rise to the Euler class, which is the image of the orientation class $u \in H^r(E, E \setminus E_0; \mathbb{Z})$ in $H^r(X; \mathbb{Z})$ under the pullback $(X, \emptyset) \to (E, \emptyset) \to (E, E \setminus E_0)$. The important thing about Euler class is that it is the vanishing locus of generic section and the Euler class of the normal bundle of Y in X is naturally identified with the self-intersection of Y in X.

Lefschetz hyperplane theorem in algebro-geometric context:

> For any smooth pair of k-varieties (Z, X) of codimension c and locally constant sheaf $\mathcal{F}$ of Λ-modules on X, there are canonical isomorphisms $H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r_Z(X, \mathcal{F})$ for all $r \geq 0$.

One corollary is that $H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F})$ is an isomorphism for $0 \leq r < 2c - 1$ and an exact sequence (the Gysin sequence):

$$0 \to H^{2c-1}(X, \mathcal{F}) \to H^{2c-1}(U, \mathcal{F}|_U) \to \dots \to H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r(X, \mathcal{F}) \to \dots$$

just by replacing H^i_Z in the LES for the pair (X, U). See also here for an explanation.

From the Gysin sequence we can calculate the etale cohomology of $\mathbb{P}^m$, see Milne, Example 16.3.

The idea of the proof is the following cohomological purity result:

Let (Z, X) be a smooth pair of algebraic varieties of codimension c. For any locally constant sheaf of Λ-modules on X, $R^{2c}i^!\mathcal{F} \cong (i^*\mathcal{F})(-c)$, and $R^r i^!\mathcal{F} = 0$ otherwise.

Here $i^!\mathcal{F}$ is defined to be the $i^*\mathcal{F}^!$ where $\mathcal{F}^!$ is the largest subsheaf of $\mathcal{F}$ with support on Z. It can be written as $\mathcal{F}^! = ker(\mathcal{F} \to j_*j^*\mathcal{F})$. The functor $i^!$ is then the right adjoint of i_*, and it is left exact and preserve injectives (since its left adjoint i_* is exact). In general, the exceptional inverse image functor (see here for the intuition behind $f^!$) is only defined at the level of derived categories. There is also an extension of the purity result to general base schemes than spectrum of a field, see the theorem of absolute purity

Stable characteristic class:

Stiefel–Whitney class (a set of topological invariants of a real vector bundle that describe the obstructions to constructing everywhere independent sets of sections of the vector bundle. Stiefel–Whitney classes are indexed from 0 to n, where n is the rank of the vector bundle)

Chern class (complex analogue of Stiefel-Whitney class)

See this thread for why there is no curvature form interpretation of Stiefel-Whitney class.

Connection to Steenrod square: It is the algebra of stable cohomology operations for mod p cohomology, see this note for an introduction.

More on Steenrod square: One perspective is that Steenrod squares remember normal bundle data (self-intersection), see this answer. Another perspective is that it measures how the cup product, while homotopy-commutative (in terms of the induced maps to Eilenberg-MacLane spaces), cannot be straightened to be actually commutative, see this answer for an explanation. For more detail see Hatcher, page 502 and this note. The idea is that a cohomology class $\alpha \in H^n(X; \mathbb{Z}/2)$ has cup product $\alpha^2 \in H^{2n}(X; \mathbb{Z}/2)$, which can be viewed as a map of $X \to X \times X \to K(\mathbb{Z}/2, 2n)$. We can extend the last map $\alpha \times \alpha$ to $S^\infty \times X \times X \to K(\mathbb{Z}/2, 2n)$ by virtue of the homological commutativity of cup product (which translates to existence of homotopy f_t from $\alpha \times \alpha$ to $T(\alpha \times \alpha)$ where T is the self-map of $X \times X$ swapping the two coordinates and since $T^2 = id$ we get a loop of maps $S^1 \times X \times X \to K(\mathbb{Z}/2, 2n)$. After choosing appropriate f_t this map will be null-homotopic so extends to $D_2 \times X \times X$ and we iterate this process.). This map has the property that (s, x_1, x_2) and $(-s, x_2, x_1)$ has the same image, so it descends to

$X \times \mathbb{RP}^\infty \to K(\mathbb{Z}/2, 2n)$ which extends α. Now we use Kunneth formula and write α as $\sum \omega^{n-i} a_i$ where ω is the generator of $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$. The a_i is defined to be $Sq^i(\alpha)$.

The intuition of Adem relations is that they come from the symmetry of $\mathbb{Z}/2 \times \mathbb{Z}/2$ by swapping the factor. They actually follow from other axioms of Steenrod square, see this paper and this short note. For more see here.

The Adem relations follows from two facts: 1. $Sq^{2n-1}Sq^n = 0$ 2. $Sq^n \mapsto Sq^{n-1}$ is a derivation. Then we have a Pascal's triangle. See this note.

Previously we have computed the cohomology ring of $K(\mathbb{Q}, n)$. With integral information, we should look at each prime. Serre showed that $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \cong \mathbb{F}_2[Sq^I \iota_n : I \text{ admissible }, e(I) < n]$. See here and Hatcher's Spectral Sequence, Section 5.1 for a proof.

The theorem implies that the admissible monomials in $\mathcal{A}_p$ are linearly independent, hence form a basis for $\mathcal{A}_p$ as a vector space over $\mathbb{Z}/p$. For if some linear combination of admissible monomials were zero, then it would be zero when applied to the class ι_n, but if we choose n larger than the excess of each monomial in the linear combination, this would contradict the freeness of the algebra.

Another consequence is that a cohomological operation commutes with suspension (i.e. the stable cohomological operation) iff it is of the form Sq^I for some I. The same holds if we replace 'stable' by 'linear' (note that a cohomological operation need not be a homomorphism).

Application of Steenrod square: If $f : S^{2n-1} \to S^n$ has Hopf invariant 1, then $[f] \in \pi^s_{n-1}$ is nonzero (Theorem 4L.2 of Hatcher).

The Adem relations implies $\mathcal{A}_2$ is generated by as an algebra by the elements Sq^{2^k}.