automorphy lifting: lecture 17-19

J'ignore • 10 Nov 2025

Let (ρ, N) be a WD representation of G_{F_v} over an algebraically closed field E. We define

local L-factor: $L((\rho, N), X) = \det(1 - X\rho(\phi_v)|_{V^{I_{F_v}, N=0}})^{-1} \in E(X)$.

conductor: $f(\rho, N) = codim_E V^{I_{F_v}, N=0} + \int_0^\infty codim_E V^{I_v^u} du$

local ϵ-factor: $\epsilon((\rho, N), \psi_v)$ depends on an additive character $\psi_v : F_v \to E^\times$ (Deligne, Tate's 'Local Constant' in Fröhlich's algebraic number fields: L-functions and Galois properties, Bushnell-Hennart)

Global L-function: $L(r, s) = \prod_{v \nmid \infty} L(WD(r|_{G_{F_v}}, |k_v|^{-s}))$ which converges to a holomorphic function on $\{s : Re(s) \geq 1 + w/2\}$. Note that $L(r_1 \oplus r_2, s) = L(r_1, s)L(r_2, s)$. and $L(r \otimes \chi_p^j, s) = L(r, s + j)$.

Example: Riemann zeta function, zeta function for elliptic curves

Remark: The global L-function determines the local L-factors for a.e. v, so by Cebotarev it determines the semisimplification of r.

Next we introduce the Gamma factor (local L-factor at ∞). For reference see Deligne.

Introduce $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. For each $\tau_\infty : F \to \mathbb{C}$, define $HT_{\tau_\infty}(r) = HT(r|_{G_{F_v}})$. Let $m_{\tau_\infty, i}(r)$ be the multiplicity of i in $HT_{\tau_\infty}(r)$. If $\tau_\infty : F \to \mathbb{C}$ factors through $\mathbb{R}$ and if $w/2 \in \mathbb{Z}$, define $m_{\tau_\infty, w/2, \pm}(r) \in \frac{1}{2}\mathbb{Z}$ by $m_{\tau_\infty, w/2, +}(r) + m_{\tau_\infty, w/2, -}(r) = m_{\tau_\infty, w/2}(r)$ and $m_{\tau_\infty, w/2, +}(r) - m_{\tau_\infty, w/2, -}(r) = (-1)^{w/2}(\dim(V^{c_\infty = 1}) - \dim(V^{c_\infty = -1}))$ where c_∞ is the image of complex conjugation in G_F. Note that these are integers rather than half integers if $m_{\tau_\infty, w/2}(r) = \dim(V) \mod 2$. This condition holds for etale cohomology of algebraic variety. Define $\Gamma_{\tau_\infty}(r, s) = \Gamma_{\mathbb{R}}(s - w/2)^{m_{\tau_\infty, w/2, +}(r)} \Gamma_{\mathbb{R}}(s - w/2 + 1)^{m_{\tau_\infty, w/2, -}(r)} \prod_{i < w/2} \Gamma_{\mathbb{R}}(s - i)^{m_i(r)} \prod_{i > w/2} \Gamma_{\mathbb{R}}(s - (w - 1 - i))^{m_i(r)}$

. If $w/2 \notin \mathbb{Z}$, then define $m_{w/2, \pm} = 0$ and proceed in the same way. Finally if τ_∞ doesn't factor through $\mathbb{R}$, then define $\Gamma(r, s) = \prod_i \Gamma_{\mathbb{C}}(s - i)^{m_i(r)}$.

For the definition of ϵ-factor at infinity and the rest of the note, see here.

A Cartan subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra equal to its own normalizer.

Fact: If $\mathfrak{g} = Lie(G)$, G reductive over $k = \overline{k}$, the Cartan subalgebras of $\mathfrak{g}$ are the Lie algebras of maximal tori.

Let $K_\infty \subset G_\infty$ (the component of $G(\mathbb{A})$ at infinity) be the maximal compact subgroup (unique in the archimedean case, see here for a proof).

Example: Let $G = GL_n$. For $F = \mathbb{Q}(\sqrt(2))$, we have $G_\infty = GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ then $K_\infty = O_n(\mathbb{R}) \times O_n(\mathbb{R})$. For $F = \mathbb{Q}(\sqrt{-2})$, we have $G_\infty = GL_2(\mathbb{C})$ then $K_\infty = U_n(\mathbb{R})$.

For $F = \mathbb{Q}$ and $G = Sp_{2n}$. Then $G_\infty = Sp_{2n}(\mathbb{R})$, and $K_\infty = U_n(\mathbb{R})$ where the embedding is given by mapping $A + iB$ to $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$.

For $F = \mathbb{Q}(i)$ and G unitary group for a nondegenerate Hermitian form of signature (a, b), then $G_\infty = U_{a,b}$ and $K_\infty = U_a \times U_b$.

The upshot is the quotient G_∞ / K_∞ are nice spaces.

Fix a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^-$, $\mathfrak{t} \oplus \mathfrak{u}$ is the Lie algebra of a Borel. Roots are the weight (generalized eigenvalues) of $\mathfrak{t}$ on $\mathfrak{u}^+ \oplus \mathfrak{u}^-$. Positive roots are the weight for the action on $\mathfrak{u}^+$.

PBW: $U(\mathfrak{g}) = U(\mathfrak{t}) \oplus \mathfrak{u}^- U(\mathfrak{g}) \oplus U(\mathfrak{g})\mathfrak{u}^+$. Let $\tilde{\xi} : Z = Z(U(\mathfrak{g})) \to U(\mathfrak{t})$ by projection. Let $T_\rho : U(\mathfrak{t}) \to U(\mathfrak{t})$ be defined by $T_\rho(f)(\lambda) = f(\lambda - \rho(\lambda))$ for $\lambda \in \mathfrak{t}$ and $f \in U(\mathfrak{t})$ thought of as polynomial in $\mathfrak{t}^*$.

Harish-Chandra isomorphism: The composite $T_\rho \circ \tilde{\xi} : Z \to U(\mathfrak{t})$ induces an isomorphism onto $U(\mathfrak{t})^W$.

Thus given any representation V of $\mathfrak{g}$ with an induced homomorphism (infinitesimal weight) $Z \to \mathbb{C}$, we get a homomorphism $U(\mathfrak{t})^W \to \mathbb{C}$, determined by a W-orbit of a homomorphism $U(\mathfrak{t}) \to \mathbb{C}$ or an element of $W \setminus \mathfrak{t}^*$, known as the Harish-Chandra parameter of V.

We say a cuspidal automorphic representation $\Pi = \otimes'_v \pi_v$ is algebraic if the HC-parameter of π_∞ lies in $W \setminus (\rho + X^*(T_\mathbb{C}))$.

Serge Lang's theorem: G_{F_v} over F_v extends to a smooth reductive group scheme G_v over $\mathcal{O}_{F_v}$ (smooth affine group scheme whose geometric fibers are connected reductive) is equivalent to G_{F_v} quasi-split and split after an unramified extension.

If T_v is a maximal torus of G_v and S_v is a maximal split subtorus of T_v. In this case $T_v = Z_{G_v}(S_v)$. Similarly we define $\mathcal{H}(T_v(F_v), T_v^0)$ where $T_V^0 = T_v(F_v) \cap K_v = T_v(\mathcal{O}_{F_v})$ and similarly for S_v.

Compute $(1_{gT_v^0} * 1_{hT_v^0})(x) = 1$ if $x \in ghT^0$ and 0 otherwise. Thus $\mathcal{H}(T_v, T_V^0) = \mathbb{C}[T_v(F_v)/T_v^0]$ and similarly for S_v. Now we just need to calculate the coset.

Note: The cocharacter group $X_*(S_v) = X_*(T_v \otimes_{\mathcal{O}_{F_v}} \overline{F_v})^{Gal_{F_v}} = X_*(T_v \otimes_{\mathcal{O}_{F_v}} \overline{k_v})^{Gal_{k_v}}$ because G_v and T_v split over an unramified extension of F_v.

Fact (reference: see Casselman's note on spherical functions): $S_v(F_v) = Hom(X^*(S_v), F_v^\times)$ and $S_v(F_v) = Hom(X^*(S_v), \mathcal{O}_{F_v}^\times)$. Thus $X_*(S_v) = Hom(X^*(S_v), \mathbb{Z}) \cong S_v(F_v)/S_v^0 \cong T_v(F_v)/T_v^0$.

The Satake map $S_{T_v}^{G_v} : \mathcal{H}(G(F_v), K_v) \to \mathcal{H}(T_v(F_v), T_v^0)$ by $S_{T_v}^{G_v}(f)(x) = \delta^{1/2}(x) \int_{U_v(F_v)} f(xu)du$.

Consider $W_{S_v} := N_{G_v}(S_v)/Z_{G_v}(S_v) \cong N_{G_v}(T_v)/T_v \cong (N_{\overline{G_v}}(\overline{T_v})/\overline{T_v})^{Gal_{F_v}}$.

Satake's theorem: $S_{T_v}^{G_v}$ induces an isomorphism from the Hecke algebra of G_v to the W_{S_v}-invariant of that of T_v, hence is commutative (not true if we choose other maximal compact).

For the rest of the note, see here.