Spectra motivation

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The motivation of spectra comes from the study of generalized cohomology theory. Note that the singular cohomology theory $H^n(X;G)$ is represented by $K(G,n)$, by a simple Yoneda calculation, see here. By Brown representability, any such cohomology theory is representable by $X \mapsto [X, E_n]$. We may think that if $\Sigma E_n \to E_{n+1}$ is an isomorphism, then we can get a generalized cohomology. But in order to show there is a isomorphism coming from the connecting map $H^n(X) \to H^{n+1}(\Sigma X)$ (the fact that connecting map goes from n to $n+1$ rather than the other way also explains why proving vanishing for higher cohomology is very useful), then it eventually boils down to showing the adjoint $\Omega E_n \to E_{n+1}$ is an isomorphism, which are not true in general (e.g. fails for spheres).

A comparably naive idea is to take the Spanier-Whitehead category $\{X, Y\} = \varinjlim [\Sigma^k X, \Sigma^k Y]$. The motivation is that homology/cohomology is a stable invariant, in the sense that if $\Sigma X \cong \Sigma Y$, then already $H^*(X) \cong H^*(Y)$, and if there exists k such that $\Sigma^k f$ is null-homotopic, then the induced map f_* on cohomology is zero. Observe $\{X, Y\}$ are naturally abelian groups, and if X is compact, then $\{X, Y\} \cong \varinjlim [X, \Omega^k \Sigma^k Y] \cong [X, \varinjlim \Omega^k \Sigma^k Y] =: [X, QY]$. The funny thing is that the wedge product now becomes a coproduct in this category, so the SW category has a biproduct which implies (formally) that the Hom sets are commutative monoids. We can further enrich the Hom set with a graded abelian group structure by definining $\{X, Y\}_k := \varinjlim [\Sigma^{n+k} X, \Sigma^n Y]$. We can further invert Σ by defining $\{\Sigma^{-n} X, \Sigma^{-m} Y\} = \varinjlim_{k-n \geq 0, k-m \geq 0} [\Sigma^{k-m} X, \Sigma^{k-m} Y]$.

Later it was realized (see e.g. Whitehead 62) that this all this is fixed by regarding the SW-category for finite CW complexes as a full subcategory on the (shifted) suspension spectra inside the larger category of spectra: the stable homotopy category (e.g. Schwede 12, chapter II theorem 7.2). As such it is the full subcategory on the finite spectra (e.g. Schwede 12, chapter II theorem 7.4).

As a corollary of Freudenthal suspension theorem, we know that If $n < 2k - 1$, then for any k-connective Y, we have an isomorphism $[X, Y] \xrightarrow{\cong} [\Sigma X, \Sigma Y]$ (The idea is to apply five-lemma to the skeleton of X.). Thus $[\Sigma^n X, \Sigma^n Y]$ stablize.

The idea to show Freudenthal suspension theorem is to use the James reduced product as a model for loop space of suspension. We say X is k-connective if the $(k-1)$-skeleton of X is contractible, so X is homotopy equivalent to $\bigvee S^k$ with higher dimensional cells attached. It is easy to show that if X is k-connective, then $X^{\wedge n}$ is nk-connective. Thus if X is k-connective, then $\widetilde{H_j}(J_{n-1}(X)) \xrightarrow{\cong} \widetilde{H_j}(J_n(X))$ for $j < nk-1$ and onto for $j = kn-1$ (since $J_n(X)$ is obtained from $J_{n-1}(X)$ by attaching $X^{\wedge n}$. Let F_n be the fiber of $J_{n-1}(X) \to J_n(X)$ (after fibrant replacement). Note that since applying Σ, we have a splitting $\Sigma J_n(X) \simeq \Sigma J_{n-1}(X) \vee \Sigma X^{\wedge n}$, so $\widetilde{H_j}(J_n(X)) \cong \widetilde{H_j}(J_{n-1}(X)) \oplus \widetilde{H_j}(X^{\wedge n})$. By using the Serre spectral sequence, we can show that $\widetilde{H_j}(F_n) = 0$ if $j < kn-1$ and $\widetilde{H_{kn-1}}(F_n) \cong \widetilde{H_{kn}}(X^{\wedge n})$. By Hurewicz we then know the same holds for $\pi_j(F_n)$. Hence from the homotopy LES associated to the fibration, we have for $j \leq kn-1$, $\pi_j(J_{n-1}(X)) \xrightarrow{\cong} \pi_j(J_n(X))$ and it is onto for $j = kn-1$. In particular, $X = J_1(X) \to J(X)$ induces isomorphism on π_j for $j < 2k-1$. But $X \to J(X)$ is the unit/counit of the loop-suspension adjunction, so by factoring $[S^j, X] \to [Sj+1, \Sigma X]$ into $[S^j, X] \to [S^j, \Omega\Sigma X] \cong [S^{j+1}, \Sigma X]$, we see that this is an isomorphism for $j < 2k-1$.

Any spectrum E is weakly equivalent to an Ω-spectrum by defining $(QE)_n := \varinjlim \Omega^k E_{n+k}$, for which Brown representability can be applied. See here

Reference:

List of cohomology theories