Hamiltonian, Noether's theorem, spherical varieties

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If we do the Legendre transform to the Lagrangian $\mathcal{L} = \mathcal{L}(q, \dot{q})$, we will swithc from $(q, \dot{q})$ to (q, p) where p is the generalized momentum $\partial\mathcal{L}/\partial\dot{q}$ and the importance about this change of variables is that p sits in a more symmetric position to q. Indeed, the Lagrange's equation will be equivalent to Hamilton's equation (The Legendre transform is involutive, so we can recover Lagrangian from Hamiltonian; though see here for the discussion that hamiltonian may not be convex).

How to start from symplectic geometry and produce Hamilton's equations: The idea is Darboux's theorem, which states in a small neighbourhood around any point on M there exist suitable local coordinates q_i, p_i s.t. the symplectic form ω becomes $\sum dp_i \wedge dq_i$. The form ω induces a natural isomorphism J of the tangent space with the cotangent space (since it is closed 2-form). If $H \in C^\infty(M \times \mathbb{R}_t, \mathbb{R})$ (think of at each time t the function H_t gives the energy at each point of M), then it gives rise to a vector field defined by $\dot{x} = J^{-1}(dH)x$. To see that this corresponds to Hamilton's equation, note that $dH = (\partial H/\partial q, \partial H/\partial p)$ and $J^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, see Hamiltonian vector field for reference.

We list two properties which is immediate:

1. A Hamiltonian H is constant along the integral curve of X_H. More generally, if H, F have zero Poisson brackets, then F is constant along the integral curve of H and vice versa (the abstract form of Noether's theorem).
2. The Hamiltonian flow preserves the symplectic form ω (Liouville's theorem).

Next we want to motivate moment map. We start with a Lie group G acting by symplectomorphism on M. Intuitively M is the phase space and elements of G are symmetries. The idea of a moment map is that it gives an embedding of the Lie algebra of infinitesimal symmetries $\mathfrak{g}$ into the Poisson algebra $C^\infty(M)$. See this answer for more details. See this for how it corresponds to the usual momentum.

The moment map is also connected to Noether's theorem, in the sense that if the action $G \times M \to M$ preserves the Hamiltonian H, then for every infinitesimal $\xi \in \mathfrak{g}$ the image under (the dual) of the moment map $J(\xi)$ is a constant of motion. See here for more details.

Motivation for spherical varieties: They are essentially the simplest kind of G-varieties, in the sense that there are only finitely many G-orbits for all its G-equivariant birational models. This is equivalent to having a dense open B-orbit for a Borel B, and also equivalent to some multiplicity-free condtions if it is quasi-projective, see here (where the motivation is to find subgroup $H \subset G$ that acts locally transitively on the flag variety G/P) and here.

Chevalley gives us an isomorphism $\mathfrak{g}//G := Spec(k[\mathfrak{g}]^G) \xrightarrow{\cong} \mathfrak{t}/W$ for G connected reductive. Let $\tilde{\psi}$ denote the composite $T_X^* \to \mathfrak{g}^* \cong \mathfrak{g} \to \mathfrak{g}//G \to \mathfrak{t}/W$. For generic $x \in T_X^*$, we can look at the irreducible components of $\tilde{\psi}^{-1}(\tilde{\psi}(x))$. More precisely set $\overline{T_X}^* := T_X^* \times_{\mathfrak{t}/W} \mathfrak{t}$. This space is no longer irreducible, but W acts transitively on the components. Let $C \subseteq \overline{T_X}^*$ be one of them. Then let $W_X := N_W(C)/Z_W(C)$, i.e. W_X is the Galois group of $C \to T_X^*$. The principal result is that W_X is always a crystallographic reflection group.

Now we insert some motivations behind flag varieties. The idea is that Borel-Weil-Bott theorem essentially says that every finite-dimensional irreducible representation arises from parabolic induction. Quoted from this mathoverflow answer,

> Taken by itself, the Borel-Weil theorem provides a somewhat concrete geometric model using line bundles on the flag variety for all finite dimensional irreducible representations of a (complex or compact) semisimple Lie group. Up to isomorphism these representations are parametrized by "dominant" characters of a maximal torus. The existence was originally an indirect consequence of work by E. Cartan and then Weyl, but the actual representations are not easy to write down. Instead, some indirect information about characters (or weight space multiplicities) was cleverly developed.

The Beilinson–Bernstein localization is a power generalization of this idea. It says we have an equivalene of categories between $U(\mathfrak{g})/ker(\chi)$-modules and $\mathcal{D} - Mod(G/B)$. The LHS can be thought of as deformation of finite-dimensional G-representation with a fixed central character $\chi : Z(U(\mathfrak{g})) \xrightarrow{\cong}\to Sym(\mathfrak{t})^{W,\rho}\mathbb{C}^{\times}$ corresponding to the weight $-\rho \in \mathfrak{t}^*$.

horospherical type: first invariant is

If $X = G/U$, then the cotangent bundle of G/U has a right action of $B/U = T$ and the irreducibel compnent of $T^{\tilde{*}}X$, but the irreducible compnents are just W, so in this case it is the trivial

Horospherical subgroup $S \subset G$ is horospherical if S contains a maximal unipotent subgroup of G, if we take the normalizer P of S then it is parabolic subgroup, and S also contains $[P, P]$. Then P/S is a torus. Proof: Let V be a rep of G stablizer of a line L is S, decompose it into dominant characters, write $L = \mu \cdot v$ where $v = v_1 + ...v_n$. The fact that $U \subset S$ means that v_i are higheset weight vectors and note that the intersection of stablizer of lines generated by v_i are parabolic. We note that $[Q, Q]$ is the set of g that actually stablize v_i for all i.

Consider $X = G/S$, can view X as living over G/P but P/S also acts on X on the right in adiition to G. If we consider the action of A. There is a map $\psi_X : \mathfrak{U}_g \otimes \mathfrak{U}_{\mathfrak{A}} \to D(X)^A$, it is called A-mondromic differential operators. Take $X = G/U$, then we get a map $U_g \otimes U_h$ to $D(G/U)^T$. By Harish chandra isomorphism we can identify

There is $f \in U_{\mathfrak{A}}$ such that if we invert f in $U_{\mathfrak{a}}$ then ψ_X is surjective. The reduction is that the rational $\psi_X \otimes_R K$ is surjective (because $D(G/S)^A$ is a finite $\mathcal{U}_{\mathfrak{g}}$-module where $R = Sym(\mathfrak{a})$.

Let $\mathcal{D}_X$ be the sheaf of differential operators on X, the pushforward $\pi_* \mathcal{D}_X$ has an action by A. Define $\mathcal{D}_{\mathfrak{a}}$ to be the A-monodromic operators. The global section of $\mathcal{D}_{\mathfrak{a}}$ is $D(G/S)^A$.

If λ is a homomorphism $\mathfrak{a} \to k$ (the base field), we can form $\mathcal{D}_\lambda := \mathcal{D}_{\mathfrak{a}} \otimes_{\mathcal{U}_{\mathfrak{a}}} k_\lambda$ where k_λ is the base field k regarded as a module over the universal envelopping algebra induced by λ. Since $\mathcal{D}_\lambda$ still has an action of $\mathcal{U}_{\mathfrak{g}}$, if we take global section then there is a map $\mathcal{U}_{\mathfrak{g}} \to H^0(G/P, \mathcal{D}_\lambda)$. The theorem is if λ is dominant, then this map is surjective.

in the case of P Borel, this is the Harish-Chandra character. The argument is similar to Bernstein-Saito polynomial by passing to the generic fiber and use some finite genenration result.

Brion-Vust (local structure theorem): There is a dense open $X_0 \subseteq X$ and a parabolic $P \subset G$ (containing a given Borel) and a levi $L \subseteq P$ such that X^0 is P-equivariantly isomorphic to $P \times^L Z$ for some L-variety Z, where $Z \cong L/L_0 \times V$ where $[L, L] \subset L_0 \subset L$ and L acts trivially on V.

Corollary: On the open X^0, the subgroups $\{B_x = Stab_B(x)\}_{x \in X^0}$ are conjugate in B. If X is spherical, this is automatic. Pick one B_0. Then there exists unique homospherical $S \subset P^- = L_0$ such that $S \cap B = B_0$ (we have $B \cap [L, L] \subset B_0 \subset B \cap L$).

Example: Let $X = GL_4/Sp_4$ the variety of nondegenerate symplectic forms. This is already spherical

Pick a generic point w which is the standard basis. Let B_0 be the stablizer of w in B, it looks like two blocks of $\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$ (which is 6 dimensional, the dimension of X). The L is $GL_2 \times GL_2$. The S should be contained in $[P^-, P^-]$ and satisfies $S \cap B = B_0$, so just take $S = [P^-, P^-]$. The A associated is $\mathbb{G}_m \times \mathbb{G}_m$.

For $GL_4/GL_2 \times GL_2$, we should take B to be the stabliezer of the flag $e_1 + e_2 \subseteq e_1 + e_4, e_2 + e_3 \subseteq e_1, e_4, e_2 + e_3$, theb B_0 is the diagonal torus with first and last entry the same and second and third the same.

Describe $T_x X = T_x(P^-/L_0) \oplus T_x(V) = \mathfrak{p}^-/\mathfrak{l}_0 \oplus T_x V$ where the decomposition of $\mathfrak{p}^- = \mathfrak{p}_u^- \oplus \mathfrak{l}_0 \oplus \mathfrak{a}_0$.

We get a projection $T_x X \to \mathfrak{a}_0$ and dualize we get injection $\mathfrak{a}_0^* \to T_x^* X \subset T^* X$. The kernel of the projection $\mathfrak{a}_0 \subset \mathfrak{l} \subset \mathfrak{p} \to \mathfrak{p}/\mathfrak{s}$ is $\mathfrak{a}_0 \cap \mathfrak{s} = 0$ so the outcome is $\mathfrak{a}^* \to \mathfrak{a}_0^* \to T_x^* X \subset T^* X$. If we fix a Cartan involution θ then we can produce a maximal commutative subalgebra $\mathfrak{a}^*$ of $\mathfrak{q}$. Let $\tilde{M}_X$ be the moment map image closure. In general $T^* X \to \tilde{M}_X$ doesn't have irreducible generic fiber. Let M_X be the normalization of $\tilde{M}_X$ in $T^* X$. The map $M_X \to \tilde{M}_X$ is finite and $T^* X \to M_X$ is integral (i.e. has irreducible generic fiber). Quotient by G we get $L_X := M_X//G$. The goal is we want to produce a finite group W_X such that $\mathfrak{a}^*//W_X \cong L_X$. Knopp prove the map $\mathfrak{a}^* \to L_X$ is canonical. Let $W_1 := N_W(\mathfrak{a})/Z_W(\mathfrak{a})$ ($\mathfrak{a} = \mathfrak{p}/\mathfrak{s}$ should be thought of as quotient of $\mathfrak{h}$). By the universal property of integral closure $\mathfrak{a}^* \to \mathfrak{a}^*//W_1$ factorize through L_X. Main result: There is a unique subgroup $W_X \subseteq W_1$ such that $\mathfrak{a}^* \to \mathfrak{a}^*//W_X \cong L_X$. Often true that $T^* X//G \cong L_X \cong \mathfrak{a}^*//W_X$

Example: If $X = GL_4/GL_2 \times GL_2 = G/K$, we can take $\theta = Ad_g$ where $g = \begin{pmatrix} I_2 & 0 \\ 0 & -I \end{pmatrix}$ so that $\mathfrak{k} = \mathfrak{g}^\theta$. The Weyl group is that of Sp_4.

https://physics.stackexchange.com/questions/351903/simple-explanation-of-why-momentum-is-a-covector