Cotangent complexes

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A simplicial commutative rings is a simplicial object in the category of commutative rings (the prototypical example is $R[X_\bullet]$, where $X_\bullet$ is a simplicial set). Observe that the singular complex of a topological commutative ring is a simplicial commutative ring, and the geometric realization of a simplicial commutative ring is a topological commutative ring. The motivation comes from Dold-Kan correspondence which says when $\mathcal{A}$ is not an abelian cateogory, we can still do 'homological algebra' by working with simplicial objects instead.

Recall given a ring map $f : A \to B$, we can form its Kahler differential Ω^1_f which is well-behaved if B is smooth over A. In general we want to resolve B by smooth A-algebras. What should be 'free' simplicial A-algebras if A is a simplicial commutative ring? See definition A.6.4.

Existence of resolution: If $f : A \to B$ is a map of classical commutative rings, then there exists a free simplicial A-algebras $\mathcal{B} \xrightarrow{\simeq} B$ and a map $A \to \mathcal{B}$ s.t. the composite is f. Intuitively the resolution is $...A[A[B]] \xrightarrow{\rightarrow} A[B]$ where we either apply f or just multiply within A. This is weakly equivalent to B by the extra degeneracy argument (same argument why the bar complex is a resolution). More generally, we can consider a monad T and the algebra over the monad. A mondad often arises from unit/counit associated to an adjunction (in this case it is the functor sending S to $A[S]$), and actually every monad appears as in this way though the adjunction is not unique. One can also think of monad as a cateogorification of idempotents, and Beck's monadicity theorem is related to the effectiveness of descent in the sense that a morphism is an effective descent morphism iff the base-change functor it induces is (co)monadic (monadicity/comonadicity of an adjunction between $\mathcal{C}$ and $\mathcal{D}$ is expressing whether we can view $\mathcal{D}$ as the category of algebras over $\mathcal{C}$ or vice versa, and applying this to the push-pull adjunction between $Sh(X)$ and $Sh(Y)$ it is exactly the effectivity of descent data).

We may now define the cotangent complex of a morphism $f : A \to B$, as invented by Quillen and developed by Illusie. Let $B \xrightarrow{\simeq} \mathcal{B}$ be a simplicial resolution of B as a free simplicial A-algebra. Then let $L_{\mathcal{B}/A}$ be the simplicial $\mathcal{B}$-module obtained by forming Kahler differentials level-wise: $L_{B/A}[n] := \Omega^1_{\mathcal{B}_n/A}$. Finally we define the cotangent complex to be $L_f := L_{B/A} := L_{\mathcal{B}/A} \otimes_{\mathcal{B}} B$.

The relative cotangent sequence is not left exact in general, the reason being $\mathcal{T}_{X/Z} \to \pi^*\mathcal{T}_{Y/Z} \to 0$ is not true in general (think of $Z = X = pt$ and Y a curve). This prompt us that there could be a cohomology theory explaining the lack of left exactness.

The relative conormal sequence also suggests we can further continue the relative cotangent sequence to the left. For a eady version of the idea of simplicial resolution first look at the naive cotangent complex.

Reference: https://math.uchicago.edu/~amathew/SCR.pdf