Katz’s new proof of RH for curves and hypersurfaces over finite fields

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Let $U_0/\mathbb{F}_q$ be an affine, smooth geometrically connected curve. Let $\mathcal{F}$ be an ℓ-adic local system or lisse $\overline{\mathbb{Q}_\ell}$-sheaf. It is given as a continuous, f.d. representation of $\pi_1^{arith}(U_0)$ over $\overline{\mathbb{Q}_\ell}$.

Define the Hasse-Weil L-function

$$L(U_0/\mathbb{F}_q, \mathcal{F}, t) := \prod_{\mathcal{P} \text{ closed points}} \det(1 - T^{\deg(\mathcal{P})} Frob_{\mathcal{P}}|_{\mathcal{F}})^{-1}$$

(a priori this lies in $1 + T\overline{\mathbb{Q}_\ell}[[T]]$). Grothendieck’s cohomological formula essentially says that we can understand this L-function with each Euler factor equal to the inverse of the characteristic polynomial of local Frobenius in terms of the characteristic polynomial of the global Frobenius $Frob_q$ acting on the compactly supported cohomology $H_c^i(U, \mathcal{F})$ where U is base change of U_0 to $\overline{\mathbb{F}_q}$.

Because U_0 is affine, its compactly supported cohomology are concentrated in degree 1 and 2. The degree 2 is the dual of $H^0(U, \mathcal{F})$ up to some shift.

The idea of Katz’s proof is that the moduli space of genus g curves are path connected (using existence of space-filling curves over finite fields), and if U_0 is an affine curve connecting C_0 and C_1 where we know RH holds for C_0, then we can hope to bootstrap and prove RH for C_1. Say $f : \mathcal{C} \to U_0$ is the curve fibration where the two end points are C_0 and C_1.

Fact: There is a closed relationship between $(R^i f_* \mathbb{Q}_\ell)_P$ and $H^i(\mathcal{C}_P, \mathbb{Q}_\ell)$ where P is a closed point of U_0. Say C_P is defined over $\mathbb{F}_{\mathbb{N}P}$, then The fundamental compatibility is that

$$\det(1 - T Frob_P | R^i) = \det(1 - T Frob_{\mathbb{N}P} | H^i(C_P, \mathbb{Q}_\ell))$$

We want to prove the local system R^1 is pure of weight one. Replacing R^1 by the one half Tate-twisted local system $R^1 f_* Q_\ell(1/2)$ on which $Frob_P$ is divided by $(q^{1/2})^{\deg(P)}$, we see that it suffices to show all eigenvalues of any $Frob_P$ have, via any field embedding $\iota : \overline{\mathbb{Q}_\ell} \to \mathbb{C}$, absolute value ≤ 1 (Because then it implies on R^1 itself, all eigenvalues of any $Frob_P$ have, via ι, absolute value $\leq \mathbb{N}P^{1/2}$; and from the functional equation the inequality is an equality).

We already know it for one closed point (corresponding to C_0). The idea is to the cohomological expression of the L-function

$$L(U_0/\mathbb{F}_q, \mathcal{F}, T) = \frac{\det(1 - TFrob_q | H_c^1(U, \mathcal{F}))}{\det(1 - TFrob_q | H_c^2(U, F))}$$

(combined with the Rankin-Selberg trick!) implies that bounds on the absolute value of the image of the eigenvalues of $Frob_q$ on H_c^2 under ι can be used to bound that of the local Euler factor (The key condition on $\mathcal{F}$ is ι-real, which implies for even tensor power $\mathcal{F}^{\otimes 2k}$, the local Euler factors are in $1 + T\mathbb{R}_{\geq 0}[[T]]$).

But for H_c^2 we have a description in terms of coinvariants. More precisely, $H_c^2(U, \mathcal{F}) = (\mathcal{F})_{\pi_1^{geom}}(-1)$ and viewing these coinvariants as a quotient representation of, the action of $(Frob_q)^{\deg(P_0)}$ is just the action of $Frob_{P_0}$ on this quotient. In other words, $\beta_{2k}^{\deg(P_0)}$ is among the eigenvalues of $Frob_{P_0}$ on $\mathcal{F}^{\otimes 2k}$.

Reference:

https://web.math.princeton.edu/~nmk/baby16.pdf

https://math.berkeley.edu/~fengt/Weil_I.pdf