Deligne-Lusztig theory

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Lang-Steinberg: If G is a connected algebraic group defined over $\mathbb{F}_q$, define the Lang map $L : G \to G$ by $g \mapsto g^{-1}F(g)$, then L is surjective (connectedness is crucial). This is also true for abelian varieties. Some famous examples are $\mathbb{G}_a$ and this reduced to Artin-Schreier exact sequence. In the case of $\mathbb{G}_m$ it becomes Kummer exact sequence.

Corollary: Let V be a variety acted by a connected group G. Let $\mathcal{O} \subset V$ be a G-orbit. Assume G, V and the action is defined over $\mathbb{F}_q$, and $\mathcal{O}$ is stable under F. Then $\mathcal{O}^F \neq \emptyset$. (Proof is that if $x \in \mathcal{O}$ then $F(x) = g^{-1}x$ for some g and use Lang-Steinberg to write $g = h^{-1}F(h)$ then $hx \in \mathcal{O}^F$.) This is specific to finite fields, e.g. if $F(x) = \frac{-1}{\overline{x}}$ then $\mathbb{P}^1(\mathbb{C})^F = \emptyset$.

Corollary: Let $H \subset G$ inclusion of algebraic groups over $\mathbb{F}_q$. Then $(G/H)^F = G^F/H^F$ where H is connected and G could be disconnected. This is because the map $G^F/H^F \to (G/H)^F$ is surjective by the previous corollary (because a H-coset is an H-orbit) and injectivity is easy. Connectedness of H is crucial because if $G = \mathbb{G}_m$ and $H = \{\pm 1\}$ then $G/H \cong \mathbb{G}_m$ by the squaring map $x \mapsto x^2$ and the map $G^F \to (G/H)^F$ can be identifies with it, which is not surjective from $\mathbb{F}_q^\times$ to $\mathbb{F}_q^\times$.

Corollary: Reductive groups over finite fields are quasi-split (note that T is F-stable$ doesn't mean T is split; it just means T is defined over $\mathbb{F}_q$ but the Galois action on it could still be nontrivial). If G is connected algebraic group over $\mathbb{F}_q$. There exists $T \subset B \subset G$ such that $F(T) \subset T$, $F(B) \subset B$. To find a F-stable Borel we just let $V = G/B$ be the variety of Borel subgroups and this is a single G-orbits stable under F (since $F(B)$ is still a Borel). The same argument applies to finding an F-stable maximal torus.

Note that for a single orbit $\mathcal{O} \subset V$, the induced action G^F on $\mathcal{O}^F$ (which is nonempty by the previous corollary) is not necessarily transitive (e.g. $\mathbb{G}_m$ acting on itself by $x \cdot v = x^2 v$).

Theorem: G^F-orbits on $\mathcal{O}^F$ are in bijection with the F-conjugacy classes of $Stab_G(x)/Stab_G(x)^\circ$ for any $x \in \mathcal{O}^F$. The map is given by $gx \mapsto g^{-1}F(g)$.

Corollary (DL, classification of F-stable maximal torus): The G^F-conjugacy classes of F-stable maximal torus is in bijection with F-conjugacy classes of $N(T)/Z(T) = W$ (in the reductive case $Z(T) = N(T)^\circ$.) The map is given by mapping $T' = gTg^{-1}$ to $g^{-1}F(g)$. Moreover, there are isomorphisms $(T')^F \cong (T)^{w\circ F}$, here w is the image $g^{-1}F(g)$.

Example: If $G = GL_n$ and T the standard diagonal maximal torus. Then F acts trivially on $W = S_n$, so the conjugacy classes of F-stable maximal torus are in bijection with (ordinary) conjugacy classes of S_n.

Let G be an algebraic group (possibly disconnected). There is a bijection between F-conjugacy classes of G to F-conjugacy classes in G/H if H is a connected normal subgroup defined over $\mathbb{F}_q$. The nontrivial direction follows form using a different $\mathbb{F}_q$-struture (essentially corresdponding to embedding of H into affine space $\mathbb{A}^n$ and use the Frobenius on $\mathbb{A}^n$) given by $z \mapsto xF(z)x^{-1}$ (by Lang-Steinberg theorem we can modify the embedding and hence it is a Frobenius structure), and apply Lang-Steinberg theorem to it. Thus if G is reductive $T = Z(T) \subset N(T)$ we get a bijection between F-conjugacy classes in $N(T)$ and F-conjugacy classes in W. Injectivity is easy and surjectivity again follows from Lang-Steinberg.

An algebraic group G can have many different $\mathbb{F}_q$-rational structures. Over $\mathbb{C}$, $GL_n(\mathbb{R})$ is the set of fixed points under conjugations, but we can use a different conjugations, e.g. $g \mapsto (\overline{g}^T)^{-1}$, then the set of fixed points are $U_n(\mathbb{C})$. They are two different real structures on GL_n.

Example: In $G = SL_2$,

$$T_s^F \cong T^{s\circ F} = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} t^{-q} & \\ & t^q \end{pmatrix} \right\} \cong \mathbb{F}_{q^2}^{N=1}$$

. In $G = GL_n$, for every partition $(n_1 \geq .. \geq n_l)$, then $T_w^F = \mathbb{F}_{q^{n_1}}^\times \times ... \times \mathbb{F}_{q^{n_l}}^\times$. But to write down the explicit embedding of T_w^F into $GL_n(\mathbb{F}_q)$ requires solving $g^{-1}F(g) = w$. If we let w be the longest element (n), then $T_w^F = \{(t_1, ..t_n) \in T_n : t_1 = t_n^q, t_2 = t_1^q ..., t_n = t_{n-1}^q\} \cong \mathbb{F}_{q^n}^\times$.

Let V be a variety over $\mathbb{F}_q$ with $F : V \to V$. Let $\sigma : V \to V$ be an automorphism such that $(\sigma \circ F)^n = F^n$ for some $n \geq 1$. Then $F' = \sigma \circ F$ is a Frob map associated to a $\mathbb{F}_q$ structure on V.

Example: If $\sigma : \mathbb{G}_m \to \mathbb{G}_m$ denotes the inversion map, then $F' = \sigma \circ F$ is a different Frobenius with $\mathbb{G}_m^{F'} = \{x : x^{q+1} = 1\}$. We have a commutative diagram.

In particular we can let $V = T$ and $\sigma \in W$ (note that $w \circ F \neq F \circ w$ in general).

Define $\sigma : GL_n \to GL_n$ sending g to $(g^T)^{-1}$. In this case $\sigma \circ F = F \circ \sigma$ and $F'^2 = F^2$. We have $GL_2^{F'}$ is called the unitary group $U_n(\mathbb{F}_q)$. If F is the usual conjugation then this the usual unitary group. Unfortunately, F' still acts trivially on W_n. The F'-stable maximal torus in the unitary group is still in bijection with partitions of n.

Consider $G = SO_{2n}$ associated to J the anti-diagonal all-one matrix. Take the maximal torus $T = (t_1, ...t_n, t_n^{-1}, ..., t_1^{-1})$. Let σ be Ad_{σ_n} where $\sigma_n \in O_{2n} \setminus SO_{2n}$ is the permutation matrix corresponding to $(n, n+1)$ (so this is not an inner automorphism of SO_{2n}). It is an outer automorphism of the Dynkin diagram of swapping the branching nodes. Since F and σ commutes, we have $F'^2 = F^2$. We call $SO_{2n}^{F'}$ the twisted special orthogonal group. The Weyl group of SO_{2n} is $S_n \ltimes (\pm 1)^n_{even}$ where ± 1 swap the i-th and $2n - i + 1$-th factor, where we only consider even number of swaps. It can be generated by $\sigma_1, ...\sigma_{n-1}$ where σ_i swaps $t_i^{\pm 1}$ with $t_{i+1}^{\pm 1}$ and σ_n swaps t_n and t_n^{-1}. Alternatively it is generated by $\sigma_1, ...\sigma_{n-1}, \sigma_n\sigma_{n-1}\sigma_n$ and F'-action just permute the last two generators.

(Steinberg) Let G_{ad} be the Chevalley group. Let $\sigma : G_{ad} \to G_{ad}$ be any outer-automoprhism of A_n, D_n, E. Then F' is a Frobenius of G_{ad} and $G_{ad}^{F'}$ is a finite simple group called the twisted Chevalley group if $|\mathbb{F}_q| \geq 5$.

A F-stable maximal torus T if $T^F \cong ((\mathbb{F}_q)^\times)^{\dim(T)}$. G is split if it has a split maximal torus.

Exercise: $U_n(\mathbb{F}_q)$ is not split. The F-stable maximal tori are T_0 and T_s where $T_0^F \cong (\mathbb{F}_{q^2}^\times)^{N=1}$ and $T_s^F = \left\{ \begin{pmatrix} x & \\ & x^{-q} \end{pmatrix} \right\} \cong \mathbb{F}_{q^2}^\times.$

For non-simply-laced Dynkin diagram like $B_2, there are exceptional $\mathbb{F}_q$-structures when $q = 2^n$ or 3^n and n odd. They are known as Suzuki and Ree groups.

Goal: For each F-conjugacy class of W_T corresponding to the F-stable maximal torus T', construct representations of G^F parametrized by (complex-valued) characters of T'.

The idea is to have a common geometric object X for which both T^F and G^F acts and if their actions commute, then we can link characters of T^F to representations of G^F. For the standard maximal torus $T \subset SL_2$ if we take $SL_2(\mathbb{F}_q)/U(\mathbb{F}_q)$ then we get the principal series.

We have seen that we can always find a F-stable maximal torus inside a F-stable Borel. But there are F-stable maximal torus that are not contained in any F-stable Borel.

DL's idea to construct such an object X is to use the Lang's map $L : G \to G$ given by $g \mapsto g^{-1}F(g)$. Note that the fibers are G^F-torsor (act by left multiplication). We now want to find a collection of fibers that are acted on by T^F.

Pick a Borel B containing T, and let U be the unipotent radical of B. Let $Y := L^{-1}(U)$. Then T^F acts by right mutiplication on Y. Actually $U \cap F(U)$ also acts by right multiplication on Y. The quotient $\tilde{X} := Y/(U \cap F(U))$ and $X := \tilde{X}/T^F$ are the so-called Deligne-Lusztig varieties.

Example: Assume $F(B) = B$, then $F(U) = U$, then $\tilde{X} = \{g \in G : g^{-1}F(g) \in U\} \subset G/U$ and it is easy to see that $\tilde{X} = (G/U)^F = G^F/U^F$ (since U is connected) and $X = G^F/B^F$. In this X and $\tilde{X}$ are zero-dimensionnal varieties so only H^0 is interesting.

Alternative description using Weyl group and twisted Frobenius structure: Pick $T_0 \subset B_0$ F-stable. Let $W = N(T_0)/T_0$ and $X = G/B_0$. For any $w \in W$, define $X_w = \{B \in X : B \sim^w F(B)\}$. Note that
$X_w = \{gB_0 : g^{-1}B_0 g \sim^w F(gB_0g^{-1}) = \{gB_0 : (gB_0, F(g)B_0) \in Y_w\} = \{gB_0 : g^{-1}F(g) \in B_0wB_0\}$
. If we pick a lift $w^\bullet$ of w, define $\tilde{X}_{w^\bullet} = \{gU_0 : g^{-1}F(g) \in U_0w^\bullet U_0\}$ and $Y_{w^\bullet} = \{g \in G : g^{-1}F(g) \in w^\bullet U_0\}$.

Proposition: 1. G^F acts on $Y_{w^\bullet}, \tilde{X}_{w^\bullet}, X_w$ on the left. 2. $T_0^{w\circ F}$ and $U_0 \cap w^\bullet U_0 w^{\bullet,-1}$ act on $Y_{w^\bullet}$ on the right. 3. $Y_{w^\bullet}/(U_0 \cap w^\bullet U_0 w^{\bullet -1}) \xrightarrow{\cong} \tilde{X}_{w^\bullet}$. 4. $Y_{w^\bullet}/(T_0^{w\circ F} \ltimes U_0 \cap w^\bullet U_0 w^{\bullet,-1}) \xrightarrow{\cong} X_w$.

There are G_F-equivariant isomorphism from $Y_{T\subset B}$ to $Y_{w^\bullet}$. The map is given by sending g to gx where $w^\bullet = x^{-1}F(x)$ and similarly for the X's. For detail see this note. The advantage of the former is that it's more canonical while the advantage of the latter is that we can do explicit equation by choosing the standard T_0, U_0.

Example: For $G = SL_2$, pick $w^\bullet = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, then $Y_{w^\bullet}$ consists of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $a^q = -b, c^q = -d, b^q = a - bu, d^q = c - du, ad - bc = 1$, which can be solved to yield $ad - db^q = bc - bd^q$ and $a^q = -b$ and $c^q = -d$, and hence $a^{q^2}c^q - a^q c^{q^2} = 1$. Taking q-th root yield the Drinfeld curve. In this case $U_0 \cap w^\bullet U_0 w^{\bullet -1} = e$, so $Y_{w^\bullet}$ is isomorphic to $\tilde{X}_{w^\bullet}$, and $X_w = \{B \in G/B_0 : B \neq F(B)\} = \mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus \mathbb{P}^1(\mathbb{F}_q)$, which is the finite field analogue of the upper and lower half-plane $H^\pm = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. The p-adic analogue is called the Drinfeld's half-plane.

In the case of unitary group $G^F = U_3(\mathbb{F}_q)$ and $w = (123)$, $X_w = \{B \in X : B \sim^w F(B)\} \cong \{v_1^{q+1} + v_2^{q+1} + v_3^{q+1} = \langle v, F(v)\rangle = 0\} \subset \mathbb{P}^2$. G^F acts on X_w since $\langle gv, F(gv)\rangle = \langle (F(g)^T)^{-1} gv, F(v)\rangle = \langle v, F(v)\rangle$. This case is done by Tate-Thompson, and the induced action on the H^1 contains interesting unipotent cuspidal representation. In the real case it is related to spherical harmonics.

To state the main theorem of Deligne-Lusztig it is better to use cohomology with compact support because it makes Lefschetz trace formula works, which holds for non-smooth varieties as well. The homology counterpart is the Borel-Moore homology, which are formed by replacing chains with locally finite chains (Borel−Moore homology is a covariant functor with respect to proper maps, e.g. $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is a counterexample. Similarly for cohomology with compact support).

We define the virtual representation $R^\theta_{T\subset B} = \sum_{i\geq 0}(-1)^i H^i_c(\tilde{X}_{T\subset B})_\theta \in R(G^F)$ and its trace $\chi_{R^\theta_{T\subset B}}$.

Main properties of $R^\theta_{T\subset B}$:

1. $R^\theta_{T\subset B} \cong R^\theta_{T\subset B'}$ for any B, B' containing T.
2. Any irreducible representation ρ of G^F appears with nonzero coefficient in some R^θ_T.
3. For most choice of (T, θ), R^θ_T is $\pm$ of irreducible representation.
4. $\langle R^\theta_T, R^\theta_T\rangle = 1$ iff $|\{w \in W_T : F(w) = w, w(\theta) = \theta\}| = 1$
5. $\langle R^\theta_T, R^{\theta'}_{T'}\rangle = |\{w \in (T \setminus N(T,T')/T')^F = W(T,T')^F = T^F \setminus N(T,T')^F : w(\theta) = \theta'\}|$
   where $N(T,T')$ is the set of intertwiners that sends T' to T'.
6. $\dim R^\theta_T = Tr(1, R^\theta_T) = \pm\frac{|G^F|_{p'}}{|T^F|}$, in particular it is independent of θ. There is an explicit way to determine the sign.

Fact: Let X/k be a variety with $\dim X = 0$, then $H^i_c(X) = 0$ if $i \neq 0$, $H^0_c(X) = \mathbb{C}[X]$. If $f \in Aut(X)$ is a permutation of X, then f^* induces the permutation representation of H^0_c.

Example: consider $T \subset B$ where B is F-stable. Then $\tilde{X}_{T\subset B} = G^F/U^F$, so $\dim = 0$. Then $R^\theta_T = \mathbb{C}[G^F/U^F]_\theta$ is the usual parabolic induction.

For the Drinfeld curve, we have $R^\theta_{T_s} = 1$ iff θ is regular, i.e. θ doesn't factor through the norm map, and $\dim R^\theta_{T_s} = \pm(q-1)$.

(Lefschetz fixed point formula) If $F : X \to X$ is the Frobenius map (in particular it is proepr so it induces maps on compactly supported cohomology), then $L(f, X) := \sum_{i\geq 0}(-1)^i Tr(f^* : H^i_c(X) \to H^i_c(X)) = |X^F|$.

Example: if $X = \mathbb{A}^n$, then the only nonzero cohomology is $H_c^{2n}(X) = \mathbb{Q}_\ell$ and $Tr(F^*) = q^n$, so $L(F, \mathbb{A}^n) = q^n$.

Let $g \in Aut(X)$ be a finite-order automorphism. Then

1. $L(g, X) = L^(g^{-1}, X)$.
2. $L(g, X) \in \mathbb{Z}$ and it is independent of ℓ.
3. (Kunneth formula) If $g \in Aut(X)$ and $g' \in Aut(X')$, then $L(g \times g', X \times X') = L(g, X)L(g', X)$.
4. If $X = \bigsqcup_{i\in I} X_i$ then $L(g, X) = \sum_{i\in I, g(X_i)=X_i} L(g, X_i)$.
5. If $g = su = us$ is the Jordan decomposition where s has order coprime to p and u has order power of p, then $L(g, X) = L(u, X^s)$. In particular, if $g = s$, then $L(g, X) = L(e, X^g) = \chi(X^g)$.
6. If $p : X \to X'$ is a surjective map with fibers isomorphic to $\mathbb{A}^n$ for some fixed n, and $g \in Aut(X)$ and $g' \in Aut(X')$ commute w.r.t. p, then $L(g, X) = L(g', X')$.
7. (Homotopy invariance) Let $G \subset Aut(X)$ be a connected algebraic group, then $g^* = id$ and $L(g, X) = L(e, X) = \chi(X)$.

Fact: $U \cap F(U)$ is affine. So $L(g, Y_{T\subset B}) = L(g, \tilde{X}_{T\subset B})$. The following lemma reduces the computation of character of $R^\theta_{T\subset B}$ to that of Lefschetz number:

Lemma: $Tr(g, R^\theta_{T\subset B}) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})L((g,t), \tilde{X}_{T\subset B})$ (because $\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}$ is a projection from V to V^θ)

Proof of independence of θ for $R^\theta_{T\subset B}$: calculate $\langle R^\theta_{T\subset B} - R^{\theta'}_{T\subset B}, R^\theta_{T\subset B} - R^{\theta'}_{T\subset B}\rangle$ (note that $\langle R^\theta_{T\subset B}, R^\theta_{T\subset B}\rangle = \langle R^{\theta'}_{T\subset B}, R^{\theta'}_{T\subset B}\rangle$)

Define the Green function by $Q_T^G := Tr(-, R_T^1)|_{G_u^F} : G_u^F \to \mathbb{C}$ (it turns out to have values in $\mathbb{Z}$). It is also the same for other character $\theta : T^F \to \mathbb{C}^\times$. In particular, the dimension R^θ_T is independent of θ, and equal to the Euler characteristic $\chi(X_w)$. Note that this depends on the Frobenius structure on the variety.

Example: For X_e, we have $X_e = \mathbb{P}^1(\mathbb{F}_q)$ and $X_s = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$. We have $\chi(X_e) = q + 1 = \dim R^\theta_{T_e}$ and $\chi(X_s) = \chi(\mathbb{P}^1) - \chi(X_e) = (1 - 0 + 1) - (1 + q) = 1 - q = (-1)(q - 1) = -\dim R^\theta_{T_s}$

Kazhdan-Springer formula for Q_T^G: Assume $char(\mathbb{F}_q) = p \gg 0$, let $\exp : \mathfrak{g}^F_{nil} \xrightarrow{\cong} G^F_u$. Let $B(-,-)$ be the Killing form. Fix a regular semisimple element $s \in \mathfrak{g}^F$ whose centralizer $Z_G(s)$ is a torus.

Let $n \in \mathfrak{g}_{nil}^F$ be a nilpotent element. Let $c \in k$ and $N_c = |\{x \in \mathfrak{g} : x = ysy^{-1}, y \in G^F, B(x,n) = c\}|$. Then $|Q_T^G(\exp(n))| = (N_0 - N_1)/|G^F|_p$, the p-primary part of $|G^F|$. When $n = 0$, $N_0 = (O_s)^F = |(G/T)^F| = \frac{|G^F|}{|T^F|}$ and $N_1 = 0$, so $|Q_T^G(e)| = \frac{|G^F|_{p'}}{|T^F|}$.

Let T of type $[w]$. Springer showed that $Q_T^G(u) = \sum_{i\geq 0}(-1)^i Tr(Fr \circ w : H_c^i(X_u) \to H_c^i(X_u))$ and $X_u = \{B \in X : uBu^{-1} = B\}$ is the famous Springer fiber and w acts on $H_c^i(X_u)$ via the Springer representation.

Example: If $G = GL_2$, $X_u = X = \mathbb{P}^1$, $W = S_2$ acts on H_c^0 by trivial and H_c^2 by sign representation. Then $Q_{T_e}^G(e) = 1 + q$ (since e acts trivially on H_c^*) and $W_{T_s}^G(e) = 1 - q$ (because s acts by -1 on H_c^2).

We have the following character formula for $x \in G^F$:

$$R_T^\theta(x) = \frac{1}{|Z_G(s)^{\circ,F}|} \sum_{g \in G^F, g^{-1}sg \in T^F} Q_{gTg^{-1}}^{Z_G(s)^\circ}(u)\theta(g^{-1}sg)$$

where $x = us = su$ are the Jordan decomposition of x. Note that $gTg^{-1} \subset Z_G(s)^\circ$ and $Z_G(s)^\circ$ is a connected reductive group stable under F (since s is)

Corollary: If s is not G^F-conjugate to T^F, then $Tr(x, R_{T\subset B}^\theta) = 0$ ($s \in T_{w'}$ and $T = T_w$ and w' is not F-conjugate to w).

Another corollary is $Q_T^G = Tr(-, R_T^\theta) = Tr(-, R_T^1)$ when restricted to G_u^F (because $Z_G(s) = G$ and $Q_{gTg^{-1}}^G = Q_T^G$ for $g \in G^F$).

A third corollary is if $s \in G_s^F$ is regular semisimple (i.e. $Z_G(s)^\circ$ is a torus) Then $Tr(s, R_T^\theta) = \sum_{g\in W_T^F} \theta(g^{-1}sg)$ for $T = Z_G(s)^\circ$ (This is because $x^{-1}sx \in T$ implies $x \in N(T)^F$). For other T the trace is zero by the first corollary.

Proof of character formula: Use the expression for trace in terms of Lefschetz number. We have the Jordan decomposition $(x,t) = (s,t)(u,1)$ and use property 5.

Fact:

1. The subgroup $B \cap Z_G(s)$ is a Borel subgroup of $Z_G(s)^\circ$.
2. If $g = su = us$ is a Jordan decomposition, then $u \in Z_G(s)^\circ$.
3. (Steinberg) If the derived subgroup $[G,G]$ is simply connected (i.e. $\mathbb{Z}\check{\Phi} = X_\bullet(T)$) then $Z_G(s)$ is connected.

E.g. If $G = GL_n$ and $s = (s_1, ...s_1, s_2, ...s_2, ...)$, then $Z_G(s) \cong GL_{n_1} \times ... \times GL_{n_k}$ is connected. If $G = PGL_2$ and $s = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, then

$$Z_G(s) = \begin{pmatrix} a & \\ & b \end{pmatrix} \sqcup \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix}$$

and so it is disconnected (and $[G, G] = PSL_2$ admits a 2-fold cover).

A semi-simple element $s \in G$ is regular if $Z_G(s)^\circ$ is a maximal torus. The set of regular semi-simple elements is dense.

Levi subgroup: Subgroup of the form $L = Z_G(H)$ for a torus H is called Levi subgroup.

Fact: Levi subgroups are connected reductive groups (unlike the case of semisimple elements). Proper Levi subgroup has positive dimensional center.

Note: $Z_G(s)^\circ$ may not be a Levi subgroup (in the case of GL_n it is). A counterexamle is $G = SP_4$, and $s = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, then $Z_G(s) = Sp(2) \times Sp(2)$ and it is not a Levi since its center is the Klein-four group (this also generalize to Sp_{2n}).

Consider $G = SO_{2n+1}$ and $s = diag(1, -1, -1, ...)$ and $Z_G(s) = O_{2n}$ and $Z_G(s)^\circ = SO_{2n}$ is not a Levi subgroup by the same reason. The Langlands dual of SO_{2n} is itself, which is not a subgroup of the Langlands dual of SO_{2n+1}, which is Sp_{2n}. It is an example of endoscopic subgroup of Sp_{2n}.

In a sense that the character of a regular semisimple element should determine the character. Though we are in the context of finite groups of Lie type we should consider regular unipotent elements as well.

For a non-semisimple element, we can still define what it means for it to be regular (dimension of centralizer is minimal). Fact: If g is regular, then $\dim Z_G(g) = \dim(T)$ for maximal torus T. Example: $u = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ is regular since its centralizer is $\begin{pmatrix} a & b \\ & a \end{pmatrix}$.

Fact (c.f. Steinberg's regular elements of semi-simple algebraic group):

1. $G_{reg} \subset G$ is open. If G is semisimple $\dim G \setminus G_{reg} \leq \dim G - 3$.
2. $G_{reg} \cap G_s$ and $G_{reg} \cap G_u$ are nonempty.
3. For any $g \in G_{reg}$, the centralizer is abelian and $\dim Z_G(g) = rk(G) = \dim(T)$.

For example: If $G = SL_2$, then $(SL_2)_{reg}$ is everything but the identity. The regular semisimple elements are conjugate to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, and regular unipotent elements are conjugate to $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$ and centralizer are the standard maximal torus and $\{\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\} \sqcup \{\begin{pmatrix} -1 & v \\ & -1 \end{pmatrix}\}$ (in particular disconnected).

Exercise: For $G = GL_n(k)$, TFAE:

1. g is regular
2. minimal polynomial of g has degree n.
3. k^n is a cyclic module over $k[x]$ where x acts by g.

Moreover, the centralizer has a concrete interpretation: $Z(g) \overset{\cong}{\leftarrow} (k[x]/(f))^\times$ given by mapping p to $p(g)$ where f is the minimal polynomial of g.

Proof of character formula: The key geometric lemma is

i. $Y^{(s,t)}_{T\subset B} = \coprod_{x \in G^F/(G_s^\circ)^F, xtx^{-1}=s^{-1}} Y^{(s,t)}_{T\subset B} \cap (xZ_G(t)^\circ) := Y^{(s,t)}_{T\subset B}(x)$;

ii. $Y^{G,(s,t)}_{T\subset B}(x) \cong Y^{Z_G(t)^\circ}_{T\subset B_t}$ where the isomorphism is given by $y \mapsto x^{-1}y$ and this isomorphism is equivariant for the action of $(Z_G(s)^\circ)^F \times T^F \overset{\cong}{\to} (Z_G(t)^\circ)^F \times T^F$ (by mapping (g,t) to $(x^{-1}gx, t)$).

By (i) and (ii), we have

$$Tr(g, R_T^\theta) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\sum_{x\in G^F/(G^\circ)^F, xtx^{-1}=s^{-1}} L(x^{-1}ux, Y^{Z_G(t)^\circ}_{T\subset B_t})$$

By doing a change of variable, this is equal to

$$\frac{1}{|T^F|}\frac{1}{|Z_G(s)^{\circ,F}|}\sum_{x\in G^F, x^{-1}sx\in T^F}\theta(x^{-1}sx)L(u, Y^{Z_G(s)^\circ}_{xTx^{-1}\subset B_s})$$

Finally, we note that $\frac{1}{|T^F|}L(u, Y^{Z_G(s)^\circ}_{xTx^{-1}\subset B_s}) = Tr(u, R^1_{xTx^{-1}\subset B_s}) = Q^{Z_G(s)^\circ}_{xTx^{-1}}(u)$.

Proof of (i): $y \in Y^{(s,t)}_{T\subset B}$ means that $syt = y$ and $v := y^{-1}F(y) \in F(U)$, which implies that $sF(y)t = F(y)$ implying $(syt)t^{-1}vt = syvt = yv$ which implies $t^{-1}vt = v$ so $v \in Z_G(t)^\circ$. By Lang's theorem, there exists $z \in Z_G(t)^\circ$ such that $z^{-1}F(z) = v$. Let $x = yz^{-1}$, then $F(x) = F(y)F(z^{-1}) = yvv^{-1}z^{-1} = yz^{-1} = x$, so $x \in G^F$ and $xtx^{-1} = yz^{-1}tzy^{-1} = yty^{-1} = s^{-1}$, hence $y \in Y^{(s,t)}_{T\subset B} \cap xZ_G(t)^\circ$.

Proof of inner product formula: The key is the orthogonality of Green functions:

$$\frac{1}{|G^F|}\sum_{u\in G_u^F} Q_T^G(u)Q_{T'}^G(u) = \frac{|W_{G^F}(T,T')|}{|T^F||(T')^F|}$$

In particular, if T is not G^F-conjugate to T' then the sum is zero. This is first proved (in the case of GL_n) by J. A. Green. Using this we compute

$$\langle R_T^\theta, R_T^{\theta'}\rangle = \frac{1}{|G^F|}\sum_{s\in G_s^F}\sum_{u\in(Z_G(s)^\circ)_u^F}\frac{1}{|Z_G(s)^{\circ,F}|^2}\sum\theta(x^{-1}sx)Q_{xTx^{-1}}^{Z_G(s)^\circ}(u)\overline{\sum\theta'(y^{-1}$$

by grouping the Q together, this is equal to

$$\frac{1}{|G^F|}\sum_s\frac{1}{|Z_G(s)^{\circ,F}|^2|T^F|^2}\sum\theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}|N_{Z_G(s)^{\circ,F}}(xTx^{-1},yT'y^{-1})|$$

Note that there is a bijection from
$\{(x,n',n)\in G^F\times N_{G^F}(T,T')\times(Z_G(s)^\circ)^F : x^{-1}sx\in T^F\}$ to
$\{(x,y,n)\in G^F\times G^F, G^F : x^{-1}sx\in T^F, y^{-1}sy\in(T')^F, n\in N_{(Z_G(s)^\circ)^F}(xTx^{-1},yT'y^{-1})\}$
by $y=nxn'$. Plug this into the sum, we have

$$\frac{1}{|G^F||T^F|^2}\sum_s\sum_{x,n'}\theta(x^{-1}sx)\overline{\theta'((n')^{-1}x^{-1}sxn')},$$

which is equal to (letting $t=x^{-1}sx$)

$$\frac{1}{|T^F|^2}\sum_{t\in T^F}\sum_{n'\in N_{G_F}(T,T')}\theta(t)\overline{\theta'((n')^{-1}tn')} = \sum_{w\in N_{G^F}(T,T')/T^F}\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)\overline{w^*(\theta')(t$$

Proof of Orthogonality of Green function:

Let $\overline{G}=G/Z$, where Z is the center. We note that $Q_T^G(u)=Q_{\overline{T}}^{\overline{G}}(\overline{u})$, since $G_u\cong\overline{G_u}$, and $X_w\cong\overline{X_w}$.

Lemma (Special case of geometric conjugacy and disjointness theorem): $\langle R_T^\theta, R_{T'}^1\rangle=0$ if $\theta\neq 1$.

Assume Theorem holds for groups of smaller dimension. In particular, it holds for $Z_G(s)^\circ$ for $1\neq s\in G_s^F$. The idea is in the character formula, we can write $\langle R_T^\theta, R_T^{\theta'}\rangle = \frac{1}{|G^F|}\sum_{u\in G_u^F}Q_T^G(u)Q_{T'}^G(u)+\sum_{w\in W_{T,T'}^F}\frac{1}{|T^F|}\sum_{t\in T^F\setminus\{1\}}\theta(t)\overline{w^*\theta'(t)}$ where the first expression is when $s=1$ and the second expression is because when $s\neq 1$, we have orthogonality by induction (since G is adjoint). Now the second term is equal to $|\{w\in W_{T,T'}^F:\theta=w^*\theta'\}|-\frac{|W_{T,T'}^F|}{|T^F|}$ by putting the term corresponding to $t=1$ back. Rearranging we see it boils down to proving $\langle R_T^\theta, R_{T'}^{\theta'}\rangle = |\{w\in W_{T,T'}^F:\theta=w^*\theta'\}|$. To avoid circular reasoning, the ingenius trick is to note that $\frac{1}{|G^F|}\sum_{u\in G_u^F}Q_T^G(u)Q_{T'}^G(u)$ doesn't depend on θ or θ'. Thus if T^F or T'^F has a nontrivial character, then we can use the lemma above to show that both of these are zero.

The remaining case is when T^F and T'^F have no non-trivial characters, i.e. $T^F = T'^F = e$. This can only happen if $\mathbb{F}_q = \mathbb{F}_2$ and T and T' aer split tori (since $\mathbb{F}_2^\times = e$). In this case $R_T^\theta \cong R_{T'}^{\theta'} \cong Ind_{B^F}^{G^F} 1$, then $\langle Ind_{B^F}^{G^F} 1, Ind_{B^F}^{G^F} 1 \rangle = \frac{|W_T^F|}{|T^F|} = |W_T|$.

In Kazhdan-Springer we see that $\pm Q_T^G(1) = \dim R_T^\theta$ can be gotten via examining the action of w and Fr on the flag variety X. Note that G/T_0 is a $U_0 \cong \mathbb{A}^\ell$ bundle on $G/B_0 = X$ and W acts on G/T_0 by right multiplication.

Fact: $H_c^{i-2d}(X)(d) \cong H_c^i(G/T_0)$, where $Fr \otimes q^d$ action on the left is taken to Fr action on the right. From this isomorphism we get the W-action on LHS. (To see the Tate twist is needed look at the example $\mathbb{A}^d \to pt$ and $H_c^{i-2d}(pt) \cong H_c^i(\mathbb{A}^d)$ is only equivariant by putting the Tate twist on $H_c^{i-2d}(pt)$)

Corollary:
$\pm Q_T^G(1) = q^{-d} \sum_i (-1)^i Tr(Fr \circ w, H_c^i(G/T_0) = q^{-d} |(G/T_0)^{Fr \circ w}| = q^{-d} \frac{|G^{Fr \circ w}|}{|T_0^{Fr \circ w}|} = q^{-d} \frac{|G^F|}{|T^F|}$

where for the top we use Lang's theorem to show that G acted by Fr is isomorphic to G acted by $Fr \circ w$. But we cannot use Lang's theorem to the bottom since W is not connected, so we get the torus T corresponding to $[w]$.

Reference:

https://www.math.columbia.edu/~wenqili/DLsem.html