automorphy lifting: Lecture 14-16

J'ignore • 22 Oct 2025

To reiterate, a $\mathbb{Q}_p$-representation of G_E is $?$ if there exists a filtered (graded if $? = HT$) E-vector space D with trivial Galois action such that $D \otimes_E B_? \to V \otimes_{\mathbb{Q}_p} B_?$ is an isomorphism (preserving Galois action and filtration). The last statement implies that $D \cong D_?(V)$.

Fact: (Tate-Sen) For every finite extension $E/\mathbb{Q}_p$ we have $(\mathbb{C}_p(i))^{G_E} \cong E$ if $i = 0$ and 0 otherwise. More generally, if $\eta : G_E \to \mathbb{Z}_p^\times$ is a character, and set $\mathbb{C}_p(\eta) := \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)$, then $(\mathbb{C}_p(\eta))^{G_E} \cong E$ if $\eta(I_E)$ is finite and zero otherwise. This follows from the characterization of $\mathbb{C}_p$-admissibility. The proof starts by proving that the study of $\mathbb{C}_p$-semilinear representations of G_E can be reduced to that E_∞-semilinear representations of G_E where E_∞ is a totally ramified $\mathbb{Z}_p$-extension of E. To from $\mathbb{C}_p$ to $\widehat{E_\infty}$ we use almost etale descent; to go from $\widehat{E_\infty}$ to E_∞ we use some decompletion process (note that if L/K is a Galois extension, we know that there is a oneone correspondence between the elements of $H^1_{cont}(Gal(L/K), GL_n(L))$ and the isomorphism classes of L-semilinear representations of dimension n of $Gal(L/K)$). Then we can define Sen's operator, which is morally '$\log_\chi(\rho)$' (indeed it is related to the Lie algebra of $\rho(G_E)$). In particular it allows us to extract the Hodge-Tate weight. For more details see Fontaine's note, chapter 3.

Faltings's version of p-adic Hodge decomposition of etale cohomology of a proper smooth variety X/E for a finite extension $E/\mathbb{Q}_p$:

$$H^i_{et}(X_{\overline{E}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{a \in \mathbb{Z}} H^{i-a}(X, \Omega^a_{X/E}) \otimes_E \mathbb{C}_p(-a)$$

Moreover, this isomorphism preserves Galois action. The theorem implies that etale cohomology is Hodge-Tate.

To show it is further de Rham (i.e. to capture the de Rham filtration not just the Hodge-Tate weight), we can tensor with B_{dR} and get an isomorphism that is compatible with filtration

$$H^i_{et}(X_{\overline{E}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^i_{dR}(X/E) \otimes_E B_{dR}$$

The only problem is that after tensoring with B_{dR} we don't know how to undo it. A lot more can be said if we know the existence of some (p-)integral model $\mathcal{X}$ of X over $\mathcal{O}_E$, i.e. a proper flat scheme $\mathcal{X}/Spec(\mathcal{O}_E)$ satisfying some good properties, e.g. smooth or semi-stable over $\mathcal{O}_E$. In the former case we can define crystalline cohomology and in the latter case we can define log crystalline cohomology (since semi-stable implies log smooth over $\mathcal{O}_E$ with certain conditions like ‘of Cartier type’). Th reason why such an integral model is useful is because of standard machinary like proper/smooth base change.

The Fontaine-Mazur conjecture states that a $\mathbb{Q}_p$-representation r of G_F that is unramified at all but finitely many places and de Rham at places above p. Then there exists projective smooth algebraic varieties X/F and integers $i \geq 0$ and j such that r is isomorphic to a subquotient of $H^i(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(j)$. All known cases involve the use of some modular curves or the higher dimensional generalizations plus some standard (often difficult) techniques to extrapolate from those cases plus some special cases with finite image. See this post for motivation behind this conjecture.

Now we define crystalline representations. Let A°_{crys} be the divided power envelope of A_{inf}. More precisely, it is the A_{inf}-subalgebra of B^+_{inf} generated by $\frac{x^m}{m!}$ for $x \in ker(\theta)$. Let A_{crys} be the p-adic completion of A°_{crys} and $B^+_{crys} := A_{crys}[1/p]$ and $B_{crys} := B^+_{crys}[1/\xi]$. There are several facts:

1. $B_{crys} \otimes_{E_0} E \to B_{dR}$ is injective, where E_0 is the maximal unramified subextension of $E/\mathbb{Q}_p$.
2. $B^{G_E}_{crys} = E_0$.
3. The induced map between graded pieces is an isomorphism.
4. The Frobenius map on A_{inf} induces $t \mapsto pt$ where t is the p-adic analogue of $2\pi i$.
5. The induced filtration on B_{crys} is not ϕ-stable. In fact, ϕ on B_{inf} doesn’t preserve $ker(\theta)$. and ϕ doesn’t extend to B_{dR}.

One key technical result we need is $(Fil^0 B_{crys})^{\phi=1} = \mathbb{Q}_p$. This will allow us to recover V from $D_{crys}(V)$. Namely, we have $V \cong V \otimes_{\mathbb{Q}_p} (Fil^0 B_{crys})^{\phi=1} \cong Fil^0(D_{crys}(V) \otimes_{E_0} B_{crys})^{\phi=1}$.

Crystalline cohomology is by design an unramified object. It takes as input a scheme X/k where $char(k) = p$ and output a module $H^*(X/W)$ over the ring of Witt vectors $W = W(k)$. Roughly speaking, crystalline cohomology of a variety X in characteristic p is the de Rham cohomology of a smooth lift of $\mathcal{X}$ to characteristic 0, while de Rham cohomology of X is the crystalline cohomology reduced mod p. The idea of crystalline cohomology, roughly, is to replace the Zariski open sets of a scheme by infinitesimal thickenings of Zariski open sets with divided power structures. The motivation for this is that it can then be calculated by taking a local lifting of a scheme from characteristic p to characteristic 0 and employing an appropriate version of algebraic de Rham cohomology.

Crystalline comparison theorem (Faltings): Let $\mathcal{X}$ be a proper smooth variety over $E/\mathbb{Q}_p$ with good reduction and let X be the reduction of $\mathcal{X}$ mod p. Then

$$H^i_{et}(\mathcal{X}_{\overline{E}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{crys} \cong H^i_{crys}(X/E_0) \otimes_{E_0} B_{crys}$$

If X does have a proper smooth model $\mathcal{X}$ over $\mathcal{O}_E$, then the p-adic etale cohomology $H^i_{et}(\mathcal{X}_{\overline{E}}, \mathbb{Q}_p)$ is crystalline. Thus bad Galois representation forces bad geometry. But there are situations where we know the Galois representation is crystalline but it is not known whether a proper smooth model exists.

Semistable representations: There exists a unique homomorphism $\log_p : \overline{\mathbb{Q}_p}^\times \to \overline{\mathbb{Q}_p}$ such that $\log_p|_{1+\mathfrak{m}}$ is the usual log, and $\log_p|_{[\overline{\mathbb{F}_p}^\times]} = 0$ and $\log_p(p) = 0$ (Iwasawa's convention).

For every $x \in 1 + \mathfrak{m}_{\mathcal{O}_{C^\flat}}$, the element $\frac{([x]-1)^j}{j} \in B^+_{inf} = W(\mathcal{O}_{C^\flat})[1/p] \subset B^+_{crys} = A_{crys}[1/p]$ lies in A_{crys} for $j \gg 0$ and converges to zero in the p-adic topology. Thus we can make sense of $\log_{crys}([x]) \in B^+_{crys}$. We extend this to all $\mathcal{O}^\times_{C^\flat}$ by setting $\log_{crys}|_{\overline{\mathbb{F}_p}^\times} = 0$. Then we have $\lambda_{crys} : \mathcal{O}^\times_{C^\flat} \to B^+_{crys}$ by composing the Teichmuller map $[]$ with $\log_{crys}$. The map λ_{crys} induces a $\mathbb{Q}$-algebra homomorphism $Sym_{\mathbb{Q}}(\mathcal{O}^\times_{C^\flat}) \to B^+_{crys}$. Let $B^+_{st} := B^+_{crys} \otimes_{Sym_{\mathbb{Q}}(\mathcal{O}^\times_{C^\flat})} Sym_{\mathbb{Q}}((C^\flat)^\times)$, which is noncanonically isomorphic to $B^+_{crys}[X]$ by matching some nonzero $y \in \mathfrak{m}_{C^\flat}$ with X. Finally define $B_{st} = B^+_{st}[1/\xi]$ nononically isomorphic to $B_{crys}[X]$.

The crystaline Frobenius ϕ mapping t to pt extends to B^+_{st} by mapping $\lambda_{crys}(x)$ to $p\lambda_{crys}(x)$. Noncanonically it is $X \mapsto pX$.

The new structure we get on B^+_{st} is that for any nonzero $y_0 \in \mathfrak{m}_{\mathcal{O}_{C^\flat}}$ such that $\lambda_{st}(y_0) = X$ so that $B^+_{st} \cong B^+_{crys}[X]$, if we set $N := v(y_0)\frac{d}{dX}$ ('monodromy operator'), then N is an endomorphism of B^+_{st} that is independent of the choice of y_0 and the isomorphism $B^+_{st} \xrightarrow{\cong} B^+_{crys}[X]$. The map N is B^+_{crys}-linear and $B^+_{crys} = (B^+_{st})^{N=0}$ and it satifies $N\phi = p\phi N$, and it is compatible with G_E-actions.

Every coset in $(C^\flat)^\times/(\mathcal{O}^\times_{C^\flat}$ is represented by some z such that $z^\# \in \overline{\mathbb{Q}_p}^\times$ (which implies $[z]/z^\# \in B^+_{dR}$ makes sense) and we can define $\lambda(z) := \log_{dR}([z]/z^\#) + \log_p(z^\#) \in B^+_{dR}$. For z such that $z^\# \in \overline{\mathbb{Z}_p}^\times$, we can define $\lambda_{crys}(z)$ in the same way. Then λ is a Galois-equivariant map from B^+_{st} to B^+_{dR} over B^+_{crys} and $B^+_{st} \otimes_{E_0} E \to B^+_{dR}$ is injective.

Similar to crystalline representations, for semistable representations V we have $V \cong Fil^0(D_{st}(V) \otimes_{E_0} B_{st})^{N=0,\phi=1}$.

To summarize $B_{crys} \subset B_{st} \subset B_{dR}$ and $grB_{dR} = B_{HT}$ and crystalline implies semistable implies de Rham implies Hodge-Tate.

Let MF_E^{ϕ} be the category of filtered ϕ-modules $(D, \phi, Fil^{\bullet})$. The tuple (D, ϕ) is isocrystal over E_0, where D is a finite-dimensional E_0-vector space with a Frobenius-semi-linear $\phi : D \to D$ (i.e. $\phi(ax) = \sigma(a)\phi(x)$ where σ is a lift of $x \mapsto x^p$). The filtration is on $D_E := D \otimes_{E_0} E$. Similarly let $MF_E^{\phi,N}$ be the category of filtered (ϕ, N)-modules equipped with E_0-linear $N : D \to D$ such that $N\phi = p\phi N$.

Let $Rep_{\mathbb{Q}_p}^{crys}(G_E)$ be the category of finite-dimensional crystalline representation of G_E over $\mathbb{Q}_p$.

Fact: The functor D_{crys} from $Rep_{\mathbb{Q}_p}^{crys}(G_E)$ to MF_E^{ϕ} is fully faithful and $V_{crys}(D) := Fil^0(D \otimes_{E_0} B_{crys})^{\phi=1}$ provides a quasi-inverse on the essential image. Similarly for $Rep_{\mathbb{Q}_p}^{st}(G_E)$.

The essential image is the so-called weakly admissible modules (a result of Colmez-Fontaine). The condition is for any subobject $D' \subset D$, we have $t_N(D') \geq t_H(D')$. Roughly speaking, it means the Newton polygon defined by ϕ (over $E_0^{nr} \subseteq \overline{E}$) is above the ‘Hodge polygon’ defined by $Fil^{\bullet}$, and the two endpoints are the same.

Fact: Subquotient of cryst/st/dR/HT are cryst/st/dR/HT. But extensions need not preserve these properties. However, if V is dR and V' and V'' are st, then V is st (Hyodo, Nekovar).

Semistable comparison: $\mathcal{X}$ semistable over $\mathcal{O}_E$ means $\mathcal{X}$ is etale locally of the form $V(t_1 t_2, ...t_n - \varpi_E)$. In this case $\mathcal{X}$ is flat over $\mathcal{O}_E$, and it is regular. Also, the special fiber $\overline{\mathcal{X}}$ is a vertical divisor on $\mathcal{X}$ with normal crosing.

We say in this case X has semistable reduction. Kato, Hyodo define log-smoothness. With this setup, we can define log-crystalline cohomology $H^i_{log,crys}(\overline{\mathcal{X}}/\mathcal{O}_{E_0})$ (we can also do $H^i_{log,crys}(\overline{\mathcal{X}}/W_n(k_E))$ for all $n \geq 1$), which is equipped with crystalline Frobenius ϕ and monodromy operator N. Roughly speaking, log structure systematically equips varieties with log poles and residues along $T = 0$ give the monodromy operator. For $\mathbb{A}^1$ we define $\Omega^{log,1}_{\mathbb{A}^1_k/k} = \mathcal{O}_{\mathbb{A}^1}\frac{dT}{T}$ and $\mathbb{A}^1_k$ and $\mathcal{O}_E$ are analogous.

If $\mathcal{X}$ smooth over $\mathcal{O}_E$, then log-crystalline cohomology agrees with usual crystalline cohomology (i.e. $N = 0$).

Theorem (Tsuji, Faltings): $H^1_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \cong H^1_{log,crys}(X/E_0) \otimes_{E_0} B_{st}$

In crystalline and semistable comaprisons, the existence of some integral model proper smooth or sst over $\mathcal{O}_E$ is a highly nontrivial condition to check. But the comparison theorem still tell us the geometry can't be better than what the Galois representations reflect.

We will say a galois represetntation $V/\mathbb{Q}_p$ is potentially crys/sst if there exists finite extension E'/E such that $V|_{G_{E'}}$ is crys/sst.

Theorem (Berger, Andre-Kedlaya-Mebkhout) dR iff potentially sst.

This doesn't imply the 'geometric potentially semistable conjecture', i.e. whether there exists finite extension E'/E such that there exists proper sst model over $\mathcal{O}_{E'}$. If we replace number field by function field of a curve, then it has been proved by Mumford. A base change from E to E' is necessary because there exists eliiptic curve with good reduction but a twist of it doesn't.

Nevertheless, de Jong shows the following: There exists finite extension E'/E and a proper sst $\mathcal{X}'/\mathcal{O}_{E'}$ such that $X' := \mathcal{X}' \otimes_{\mathcal{O}_{E'}} E'$ is an alteration of $X \otimes_E E'$, i.e. there exists proper generically finite surjective map from X' to $X \otimes_E E'$ which implies that BAKM's result using some generic etaleness and excision technique and induction argument plus the fact that dR extension of sst representation is sst.

Weil-Deligne representation: Let V be a G_E-representation over $\mathbb{Q}_p$ that is dR. Let E' be a finite Galois extension such that $V|_{G_{E'}}$ is sst. Consider $D_{st,E'}(V) = (V \otimes_{\mathbb{Q}_p} B_{st})^{G_{E'}}$. This is an E_0'-vector space with an action of $Gal(E'/E)$ and a semilinear ϕ and a nilpotent linear operator N such that $\phi N = pN\phi$. Define a linear action $\rho : W_E \to Aut_{E_0'}(D_{st,E'}(V))$ sending g to $\overline{g}\phi^{-a} = \phi^{-a}\overline{g}$ for each $g \in W_E$ with image $\overline{g}$ in $Gal(E'/E)$ and $x \mapsto x^{p^a}$ in G_{k_E}. Since $I_{E'}$ acts trivially, this is a continuous representation using discrete topology on the target. Then we take any $\tau_0' : E_0' \to \overline{\mathbb{Q}_p}$, and set $WD(V)_{\tau_0'} := (\rho, N) \otimes_{E_0', \tau_0'} \overline{\mathbb{Q}_p} \cong D_{st,\tau_0'}(V|_{G_{E'}})$, which is unique up to isomorphism. Set $WD(V) := WD(V)_{\tau_0'}$ for any τ_0'.

Fact/example: $WD(\chi_p) \cong \overline{\mathbb{Q}_p}$ where $\rho|_{I_E} = 1$ and $\rho(Frob) = \#k_E^{-1}$ and $N = 0$. If $\psi : G_E \to \overline{\mathbb{Q}_p}^\times$ is a Galois character then $WD(\psi) = (\psi|_{W_E}, N)$ with $N = 0$. For 1-dimensional representation, we also have semistable iff crystalline since there is no nonzero 1-dimensional nilpotent matrix.

Compatible system: A weakly compatible system $R = \{\rho_{v,p}\}$ is a collection of n-dimensional representations of G_F and isomorphism $\eta : \overline{\mathbb{Q}_p} \to \mathbb{C}$ such that 1. $\rho_p : G_F \to GL_n(\overline{\mathbb{Q}_p})$ is semistable and continuous with respect to the p-adic topology on $\overline{\mathbb{Q}_p}$. 2. They have the same Hodge-Tate weights and the same Frobenius-semisimple WD-representations (which is unramified) across all but finitely many places.

R is said to be strongly compatible if $WD_v(R)$ exists and
$\eta \circ WD(\rho_v|_{G_{F_v}})^{ss} \cong WD_v(R)$ for all finite places.

R is irreducible if $\rho_{p,v}$ is irreducible for all (p, v) and it is strictly pure of weight $w \in \mathbb{Z}$ if for all but finitely places with $WD_v(R)$ defined and for all eigenvalues α of $\rho_v(\phi_v)$ where ϕ_v is any lift of the geometric Frobenius we have $\alpha \in \overline{\mathbb{Q}} \subset \mathbb{C}$ and all the Galois conjugate of α is $|\#k_v|^{w/2}$. In this case N_v is force to be zero because of the relation between ϕ_v and N_v.

More generally, we say a WD-rep (ρ, N) of F_v over $\mathbb{C}$ is mixed if there exists an increasing filtration $Fil^W_\bullet$ on ρ by WD-subrep such that $Fil^W_i = 0$ for $i \ll 0$ and $Fil^W_i = \rho$ for $i \gg 0$ and the graded piece Fil^W_i / Fil^W_{i-1} is strictly pure of weight i. The monodromy N will map Fil^W_i to Fil^W_{i-2}. We say a WD-representation (ρ, N) of F_v is pure of weight w if it is mixed and if for every $i \in \mathbb{Z}_{>0}$, then $N^i : gr^W_{w+i} \to gr^W_{w-i}$ is an isomorphism. In general, any nilpotent monodromy operator of
N defines (up to degree shifting a ‘mondromy filtration’ with symmetric properties. The condition is that the weight filtration coincides (up to degree shifting) with the monodromy filtration.

we say R is geometric if there exists projective smooth variety X/F and $F \to \mathbb{C}$ integer $m \geq 0$ and j and a subspace $V \subset H^m(X^{an}_{\mathbb{C}}, \overline{\mathbb{Q}})$ such that for every prime p and $\eta : \overline{\mathbb{Q}_p} \xrightarrow{\cong} \mathbb{C}$ the pullback V_p of $V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ via the comparison between etale cohomology and singular cohomology is a subrepresentation of G_F and $\rho_{p,v} \cong V_{p,v}(j)$.

The definitions are meaningful because for places v having good reduction the representation $V_{p,v}(j)$ are unramified and any of them are de Rham. Any $V_{p,v}(j)$ are pure of weight $m - 2j$ (as global Galois representation so not touching the problem of bad reduction; use projecrive smooth models over $\mathcal{O}_{F_v}$ at all but finitely many places which gives $H^i_{et}(X_{\overline{F_v}}, \overline{\mathbb{Q}_p}) \cong H^i_{et}(\mathcal{X}_v \otimes_{\mathcal{O}_{F_v}} \overline{k_v}, \overline{\mathbb{Q}_p})$ and then use Weil conjecture for RHS over finite field k_v to show that $V_{p,v}(j)$ is unramified and $\eta \circ WD(V_{p,v}(j)|_{G_{F_v}})$ is strictly pure of weight $m - 2j$.)

At places v of F when we don’t know X has good reduction at v, Deligne’s weight-monodromy conjecture still predicts that the above is pure (but not necessarily strictly pure) of weight $m - 2j$ when $p \neq char(k_v)$. If $p = char(k_v)$ with X having sst model over $\mathcal{O}_{F_v}$ there are still analogues of weight-monodromy conjecture with log-crystalline cohomology (based on semistable comparison) and once we have the weight filtration (using mokrane’s work) we can compare it with the monodromy filtration.

(Tate conjecture) The cohomology of analytification $H^m(X_{\mathbb{C}}^{an}, \overline{\mathbb{Q}}) \cong \oplus_j V_j$ with the following property:

1. For each prime p and each embedding of α of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$, the pullback of $V_{j,\alpha}$ of $V_j \otimes \overline{\mathbb{Q}_p}$ under comparison isomorphism is an irreducible subrepresentation of G_F over $\overline{\mathbb{Q}_p}$.
2. For each j and each place v of F, there is a WD-representation $WD_v(V_j)$ of G_{F_v} of over $\overline{\mathbb{Q}}$ and the base change of $WD_v(V_j)$ to $\overline{\mathbb{Q}_p}$ is isomorphic to $WD(V_{j,\alpha}|_{G_{F_v}})$.
3. There is a tuple of multi-sets of integers $HT(V_j)$ indexed by $\tau : F \to \overline{\mathbb{Q}}$ such that for every $\tau_v : F_v \to \overline{\mathbb{Q}_p}$ with $\tau_v|_F = \alpha \circ \tau$, we have $HT_{\tau_v}(V_{j,\alpha}|_{G_{F_v}}) = HT_\tau(V_j)$ and the multiplicity of $i \in \mathbb{Z}$ is $\dim_{\mathbb{C}}((V_j \otimes \mathbb{C}) \cap H^{m-i}(X_{\mathbb{C}}, \Omega^i_{X_{\mathbb{C}}/\mathbb{C}}))$ using Hodge decomposition and GAGA.

If we ignore 1 then 2 is known up to Frobenius-semisimplification for all but finitely many places v and 3 is known (Katz-Messing).

Conjecture:

1. If $r : G_F \to GL_n(\overline{\mathbb{Q}_p})$ is semisimple and algebraic, then r is part of a weakly compatible system.
2. A weakly compatible system is strongly compatible.
3. An irreducible weakly compatible system R is geometric and pure of weight $\frac{2}{\dim R} \sum_{h \in HT_\tau} h$ for any $\tau : F \to \mathbb{C}$.

Conjecture 2 and 3 are known when $n = 1$ (Serre's green book).