Spectral sequence

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The goal is to lift information (aka cohomology) from the associated graded complex $\{gr_n^\bullet\}_{n\in\mathbb{Z}}$ to that of the original complex. The first approximation is $H^*(gr_n^\bullet)$. The cycles Z_1 are what lands in a lower layer (or upper index plus one) of the filtration. Thus it is natural to let Z_r be the elements that drop r levels down the filtration, and Z_∞ be the (decreasing) intersection of all of them. The boundary are the image of elements that comes from r levels above the filtration, and B_∞ be the increasing union of all of them. Let $E_r = Z_r/B_r$. Note that E_r has a differential, but what is the grading? Note that the filtration number increase by r but the grading on the complex only increase by 1. So to make things more symmetric we define $E_r^{p,q} = F^pC^{p+q}$ so then $E_r^{p,q} \to E_r^{p+r,q-r+1}$. This grading is natural if C is the total complex of some double complex.

A more general way in which spectral sequence arise is via exact couples. Some spectral sequences cannot be constructed from filtered complexes, see this discussion. The disadvantage is that it can be the case that our objects of study are not explicitly filtered or do not come from a filtered differential object. In this section we present another general algebraic setting, exact couples, in which spectral sequences arise. The ease of definition of the spectral sequence and its applicability make this approach very attractive. Unlike the case of a filtered differential graded module, however, the target of the spectral sequence coming from an exact couple may be difficult to identify.

It is easy to see for double complexes concentrated in upper left quadrant, $E_r^{p,q}$ stablize as long as $r > p, q+1$. For Serre's spectral sequence we use pullback of a cellular filtration of the base B and get a filtration of $C_*(E)$. The theorem is that there is a filtration on $H_n(E;R)$ with associated graded $\oplus_{p+q=n}E_\infty^{p,q}$. But we still need to solve some extension problem over R if we want to determine $H_*(E;R)$.

Example:

For the path space fibration $\Omega B \to P_b B \to B$ (B simply connected) Since the path space is contractible, we know what the spectral sequence is converging to, and we immediately get $E_2^{0,1} = E_\infty^{0,1} = 0$. Similarly, $0 = E_\infty^{3,0} = ker(d_3)$ implies $d_3 : \mathbb{Z} = E_2^{3,0} = E_3^{3,0} \to E_3^{0,2}$ is injective. Using this we can show that $H_q(\Omega S^3) \cong H_{q+2}(\Omega S^3)$ and is 0 for q odd and $\mathbb{Z}$ for q even.

The same argument shows $H_q(\Omega S^n) \to H_{q+n-1}(\Omega S^n)$ and is 0 for q not congruent to 0 mod $(n-1) and $\mathbb{Z}$ otherwise. More generally, if the base is S^n, then we get the Wang sequence, which says we have a filtration

$$0 \to E_\infty^{0,p+q} \to H_{p+q}(E) \to E_\infty^{n,p+q-n} \to 0.$$

As an application, we prove Hurewicz theorem: If X is a space and $\pi_r(X,x) = 0$ for every $r < k$, then $\pi_k(X,x) \to H_k(X,\mathbb{Z})$ is abelianization.

If $k = 1$, this is Poincare's theorem. If $k \geq 2$, we look at the loop-path fibration $\Omega X \to P_x X \to X$. Note that $\pi_r(\Omega X, x) = 0$ for $r < k-1$, so by induction hypothesis, we have $\pi_{k-1}(\Omega X) \cong H_{k-1}(\Omega X)$. It remains to show $H_{k-1}(\Omega X) \cong H_k(X)$.

Aside: Serre exact sequence. From the vanishing of the terms in $0 < p < \ell$ and $0 < q < k$ of the E_2-page of the Serre spectral sequence, we see that for the differential starting from $E_r^{p,0}$ to be nonzero, we must have $r \geq k+1$ and $p \geq r + \ell$. Thus if $p \leq k + \ell$, there is only one possibly non-zero differential on $E_r^{p,0}$. Similarly, for the differential $E^{r,q+1-r} \to E_r^{0,q}$ to be nonzero, we must have $r \geq \ell$ and $q + 1 - r \geq k$. Thus if $q < k + \ell - 1$, then there is only one possibly non-zero differential to $E_r^{0,q}$. In these two segments, we have $E_\infty^{p,0} = ker(d_p)$ and $E_\infty^{0,q} = coker(d_{q+1})$. If $n < k + \ell - 1$, we have a SES

$$0 \to coker(d_{n+1}) \to H_n(E) \to ker(d_n) \to 0$$

. Note that $d_n : H_n(B) = E_n^{n,0} \to E_n^{0,n-1} = H_{n-1}(F)$. Splicing these together we get LES ... $\to H_{n+1}(B) \to H_n(F) \to H_n(E) \to H_n(B) \to H_{n-1}(F) \to \dots$. Transgression: This is the last possible differential $d_k : E_k^{k,0} \to E_k^{0,k-1}$. There is a geometric constructino of this differential: If we look at the LES for the pair (E, F), we get

$$H_k(F) \to H_k(E) \to H_k(E,F) \to \dots,$$

Since the pair maps to (B, b), it induces a map of LES. It includes $p_* : H_k(E,F) \to H_k(B,b)$. We have a partially defined map $\partial \circ (p_*)^{-1} : Im(p_*) \to \text{quotient of } H_{k-1}(F)$. Serre shows the transgression is this map. This can be used to show that the map $H_k(X) \xrightarrow{\cong} H_{k-1}(\Omega X)$ from the Serre exact sequence is the same as the one that we need to prove Hurewicz theorem. An immediate corollary is that if X is simply connected and $H_r(X) = 0$ for $r < k$, then $\pi_r(X) = 0$ (simply connectedness is essential since π_1 could be a perfect group). Take the inverse image of A^5 in SO^3 in SU_2, it is called the binary icosahedral group $BIcos$ and it is perfect, so $\pi_1(S^3/BIcos) = BIcos$ is a counterexample. Another corollary is if X is simply connected and CW and $H_*(X) = 0$ for all X, then X is contractible. Similarly we have the homology Whitehead theorem, which is very useful in practice. The spaces with all trivila homology groups are called acyclic spaces. Another corollary is we can compute the n-th fundamental group of S^n. We can also construct the Moore space $M(A, n)$ with n-th reduced homology group A and zero otherwise. The idea is to present A as the cokernel of some $\oplus_{I_0}\mathbb{Z} \to \oplus_{I_1}\mathbb{Z}$ and let $M(A, n)$ be the mapping cone of $\oplus_{I_0}S^n \to \oplus_{I_1}S^n$.

Eilenberg-Maclane spaces $K(\pi, n)$: They are spaces with only one non-trivial homotopy group. There are multiple techniques for constructing higher Eilenberg–MacLane spaces. One of which is to construct a Moore space $M(A, n)$ for an abelian group A. Note that the lower homotopy groups $\pi_{i<n}(M(A, n))$ are already trivial by construction. Now iteratively kill all higher homotopy groups by successively attaching cells of dimension greater than $n + 1$, and define $K(A, n)$ as direct limit under inclusion of this iteration. If we attach an $(n + 1)$-cell to X via an attaching map $f : S^n \to X$, the k-th homotopy groups for $k < n$ are unchanged. Moreover, we have $\pi_n(Cf) \cong \pi_n(X)/(f)$ and $H_n(Cf) \cong H_n(X)/(f)$.

We also have the cohomological version of Serre spectral sequnce and there is a cup product on the r-th page for every r. As an example, consider $\Omega S^3 \to PS^3 \to S^3$ and note that $E_2^{p,q} = H^p(S^3; H^q(\Omega S^3) \cong H^p(S^3) \otimes H^q(\Omega S^3)$. Let $i_2 = E_2^{0,2}$. Writing $i_2^2 = ai_4$ for some integer $a \in \mathbb{Z}$, we can figure out $a = 2$ by hitting both sides with the differential d_3. More generally we can use it to show that $H^*(\Omega(S^{2n+1})) = \Gamma(i_{2n})$ and $H^*(\Omega S^{2n}) = E(i_{2n-1}) \otimes \Gamma(i_{2n-2})$ where Γ and E are divided power algebra and exterior algebra respectively. Combining this with the homology whitehead theorem we deduce $J(S^n) \simeq \Omega S^{n+1}$. More generally for a connected CW complex X, the James reduced product $J(X)$ has the same homotopy type as $\Omega\Sigma X$, the loop space of the suspension of X.

Another application is to calculate the cohomology of $K(\mathbb{Q}, n)$. The result is for symmetric algebra for n even and exterior algebra for n odd. First we need to get a concrete model of $K(\mathbb{Q}, 1)$. One explicit model is the mapping telescope defined by $S^1 \to S^1 \to S^1 ...$ where the maps are multiplication by $2, 6, ...n!, ...$. Using homotopy groups of filtered homotopy colimits, we can show this is a $K(\mathbb{Q}, 1)$. Now look at the $K(\mathbb{Q}, 1) \simeq \Omega K(\mathbb{Q}, 2) \to PK(\mathbb{Q}, 2) \to K(\mathbb{Q}, 2)$. Look at the Serre spectral sequence, we know everything on the first column (the cohomology of the fiber). From this we can inductivelly figure out the E_2 page and also the ring multiplication, the conclusion being $H^*(K(\mathbb{Q}, 2); \mathbb{Q}) = \mathbb{Q}[y_2]$ where $y_2 = E_2^{2,0}$. By induction on n we can show that for n even we have two rows and for n odd we have two columns.

Let $S^{2n+1} \to K(\mathbb{Q}, 2n + 1)$ be an element representing $1 \in \pi_{2n-1}$. This induces a map on rational cohomology and it turns out to be an isomorphism by the above calculation of $K(\mathbb{Q}, 2n + 1)$. Serre shows something more, $S_{\mathbb{Q}}^{2n+1} \simeq K(\mathbb{Q}, 2n + 1)$, i.e. the rational homotopy group of odd dimensional sphere is $\mathbb{Q}$ at degree $2n + 1$ and 0 otherwise. It turns out that all the nontorsion elements in the homotopy groups of spheres are detected either by degree or by the Hopf invariant. See Hatcher's Spectral sequence Theorem 5.22 for a proof. The idea is to find a space X admitting a fibration $X \to S_n$ with all reduced rational cohomology zero (The idea is that X should be close to S^n but with n-th homotopy group killed using $K(\mathbb{Q}, n)$), which then implies $\pi_n(X) \otimes \mathbb{Q} = 0$ for all n by Hurewicz theorem. From the LES for homotopy groups for fibration we then deduce $\pi_i(S_n)$ is finite for $i > n$.

To quote from wikipedia:

The method of killing homotopy groups", due to Cartan and Serre involves repeatedly using the Hurewicz theorem to compute the first non-trivial homotopy group and then killing (eliminating) it with a fibration involving an Eilenberg–MacLane space. In principle this gives an effective algorithm for computing all homotopy groups of any finite simply connected simplicial complex, but in practice it is too cumbersome to use for computing anything other than the first few nontrivial homotopy groups as the simplicial complex becomes much more complicated every time one kills a homotopy group. The Serre spectral sequence was used by Serre to prove some of the results mentioned previously. He used the fact that taking the loop space of a well behaved space shifts all the homotopy groups down by 1, so the nth homotopy group of a space X is the first homotopy group of its $(n-1)$-fold repeated loop space, which is equal to the first homology group of the $(n-1)$-fold loop space by the Hurewicz theorem. This reduces the calculation of homotopy groups of X to the calculation of homology groups of its repeated loop spaces. The Serre spectral sequence relates the homology of a space to that of its loop space, so can sometimes be used to calculate the homology of loop spaces. The Serre spectral sequence tends to have many non-zero differentials, which are hard to control, and too many ambiguities appear for higher homotopy groups. Consequently, it has been superseded by more powerful spectral sequences with fewer non-zero differentials, which give more information

Reference:

A guide to spectral sequence

Sepctral sequencces that everyone should know

nLab

Bockstein spectral sequence

https://mathoverflow.net/questions/45036/spectral-sequences-opening-the-black-box-slowly-with-an-example

https://www.math.ucla.edu/~mikehill/Teaching/Math885/Lecture12.pdf https://www.math.ucla.edu/~mikehill/Teaching/Math885/Lecture08.pdf

https://math.stackexchange.com/questions/50377/homotopy-groups-of-s2

https://www.college-de-france.fr/media/jean-pierre-serre/UPL7235285843586540944_Serre_The_se.pdf