Newton polygon and non-archimedean holomorphic functions

J'ignore • 22 Oct 2025

The following is the fundamental theorem about Newton polygon:

https://en.wikipedia.org/wiki/Newton_polygon#Main_theorem

In some sense it tells us precisely the valuation of the roots from the valuation of the coefficient of the polynomial. Note that we use the un-normalized valuation here (so that $v_K(x)$ doesn't depend on K).

Definition of holomorphic functions on an annulus: We can define it using Newton polygon, but it seems in the case of closed intervals we should be more careful (it seems the criterion should be for bounded holomorphic functions rather than holomorphic if I cis not open).

Example: $f(x) := \log(1+x)$ is holomorphic on the open unit disk. But it is not bounded because when $x \to -1$ (or $v(x) = \mu \to 0$) we have $V(f, \mu) = \min_{n\in\mathbb{Z}} v(a_n) + n\mu \to -\infty$. Note that the zeroes of $f(x)$ are $\zeta_{p^n} - 1$ (follows from $\log(xy) = \log(x) + \log(y)$) and they all live inside the open unit disk. Note that this is actually a criterion that determines whether a holomorphic function is bounded, see Corollary to Theorem 3.2. (but note that if K is not discretely valued, then it need not be the case that $N(f)$ above a line of slope $-r$ and $-s$ implies that it has finitely many segment of slope in (r, s))

The following roughly says that taking completion is compatible with Galois theory.

> If F is a complete p-adic field, and if $F \subset K \subset \overline{F}$, then $(\widehat{\overline{F}})^{G_K} = \widehat{K}$

Main reference: http://faculty.bicmr.pku.edu.cn/~ruochuan/2020summer/Xiao-Liang-1.pdf