James reduced product construction

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James reduced product $J(X)$ is the free monoid generated by elements of X in the category of topological spaces. We would like to find out its homology. Note that $J_i(X)$ is obtained from $J_{i-1}(X)$ by the following pushout diagram.

$$\begin{tikzcd}
(X^{i-1})^{\vee i} \arrow[r] \arrow[d, hook] & J_{i-1}(X) \arrow[d, hook] \\
X^i \arrow[r] \arrow[d, two heads] & J_i(X) \arrow[d, two heads] \\
X^{\wedge i} \arrow[r] & X^{\wedge i}
\end{tikzcd}$$

Thus we see that $J_i(X)$ is $J_{i-1}(X)$ with $X^{\wedge i}$ attached. To compute the homology of $J_i(X)$ inductively, we note that if we take (reduced) suspension of this diagram (which remains a pushout since Σ being left adjoint preserves colimit), the vertical column splits since $\Sigma(X \times Y) \cong \Sigma(X) \vee \Sigma(Y) \vee \Sigma(X \wedge Y)$. Thus $\Sigma J_i(X) \cong \Sigma J_{i-1}(X) \vee \Sigma(X^{\wedge i})$. Thus we have

$$H_*(J(X)) \cong H_{*+1}(\Sigma J(X)) \cong H_{*+1}(\bigvee_i \Sigma(X^{\wedge i})) \cong \bigoplus_i H_*(X^{\wedge i}) \cong \bigoplus_i \tilde{H}_*(X)$$

The last isomorphism follows from Kunneth and $H_*(X/A) \cong H_*(X, A)$ if $A \to X$ is a cofibration (which implies that $H_*(X \wedge X)$ is the quotient of $H_*(X \times X)$ by $H_*(X \times \{*\}) \oplus H_*(\{*\} \times X)$).

To compute the cohomology ring of $J(S^{2k})$, note that from its cellular structure we know that $H^*(J(S^{2k}))$ has exactly one $\mathbb{Z}$ at degree $2ki$ for every $i \geq 0$. To figure out the ring structure, let $e_{2ki} \in H^*(J(S^{2k}))$ be a generator of the degree $2ki$ component of $H^*(J(S^{2k}))$. We first compute e_{2k}^i for any $i \geq 0$. Note that this is a multiple of e_{2ki}, say $e_{2k}^i = c_i e_{2ki}$ for some $c_i \in \mathbb{Z}$. Let $\pi : (S^{2k})^i \to J_i(S^{2k})$ be the attaching map, which induces $\pi^* : H^*(J_i(S^{2k})) \to H^*((S^{2k})^i)$. We will compute c_i via relating the image of e_{2k}^i and e_{2ki} under π^*. Note that by Kunneth the cohomology ring of $H^*((S^{2k})^i)$ is $E(a_{2k,j}, j = 1, ...i)$, the exterior algebra with i generators at degree $2k$. It is easy to see that the image of e_{2ki} under π^* is $\prod_{j=1}^i a_{2k,j}$. To compute $\pi^*(e_{2k})$, note that the preimage of $S^{2k} \subset J_i(S^{2k})$ under π is $(S^{2k})^{\vee i}$, and it is easy to see from this that $\pi^*(e_{2k}) = \sum_{j=1}^i a_{2k,j}$. Thus

$$\pi^*(e_{2k}^i) = (\sum_{j=1}^{i} a_{2k,j})^i = c_i \prod_{j=1}^{i} a_{2k,j}$$

Since $a_{2k,j}^2 = 0$ and $a_{2k,j}$ pairwise commute, we deduce that $c_i = i!$.

From knowing $e_{2k}^i = i! e_{2ki}$ we show $e_{2ki} e_{2kj} = \binom{i+j}{i} e_{2k(i+j)}$ easily. This implies the (integral) cohomology of $J(S^{2k})$ is the divided power algebra on $2k$-dimensional class.

Reference: https://en.wikipedia.org/wiki/James_reduced_product