Operads

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Let $\mathcal{C}$ be a symmetric monoidal cateogory. Let X be an object and let $End_X(n) = Hom_{\mathcal{D}}(X^{\otimes n}, X)$. There is an action of S^n on $End_X(n)$ (transposition acts by the braiding, which is ok since it is symmetric; we also need to check $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $i > j+1$ and $(\sigma_i \sigma_{i+1})^3 = 1$ which follows from the braiding axiom)

There is an insertion operation: for every $n, m \in \mathbb{N}$ and $i \in \{1, ..., m\}$, $\circ_i : End_X(n) \times End(m) \to End(n+m-1)$. This is the prototypical example of an operad.

Sometimes $End_(n)$ has more structure than just a set, can ask that $\mathcal{C}$ is enriched over another monoidal category $\mathcal{D}$, i.e. $Hom_{\mathcal{C}}(A, B)$ is an object in $\mathcal{D}$, for more detail see here. We also want to take tensor product of an object in $\mathcal{C}$ and $\mathcal{D}$ and also define $Hom(A, B)$ as objects in $\mathcal{C}$ satisfying various axiom. For example, if $\mathcal{D}$ is a subcategory of $\mathcal{C}$, e.g. the category of vector space is cotensored over that of sets by defining $X \otimes V = \oplus_{x \in X} V$.

Let $\mathbb{S}$ be the groupoid of finite sets. The skeletons are $\mathbb{Z}_{\geq 0}$ and morphisms are from an object to itself and equal to S_n. A symmetric sequence is a $\mathbb{S}$-module in a category $\mathcal{C}$ is a functor $\mathcal{O} : \mathbb{S} \to \mathcal{C}$, determined up to natural equivalence by restriction to sk, so it is nothing but a colletion of objects $\mathcal{O}(n)$ with an action of S_n on $\mathcal{O}(n)$. An operad $\mathcal{O}$ in C is a symmetric sequence with maps, for any I, J finite sets, $i \in I$, we have an morphism $\circ_i^{I,J} : \mathcal{O}(I) \otimes_{\mathcal{C}} \mathcal{O}(J) \to \mathcal{O}(I \setminus \{i\} \sqcup J)$, and it should be natural in $(i, I), J$ (equivariant condition), which are associative: 1. Plugging in two different slots of I commute 2. Plugging in I then in J is the same as first plugging I into J then plugging into I. There is also a unital axiom.

Example: if $\mathcal{D}$ is enriched, cotensored over $\mathcal{C}$, then $\{End_X(n)\}$ is an operad. If A is a finite set, then $End_X(A) = Hom(X^{\otimes A}, X)$ where $X^{\otimes A}$ is the tensor product of X $|A|$ times, each copy labelled by elements of A.

An algebra structure on an object X over an operad $\mathcal{O}$ is a morphism $\mathcal{O} \to End_X$ between operads. If there is a hom-tensor adjunction then this is saying $\mathcal{O}(n) \otimes_{S_n} X^{\otimes n} \to X$.

Examples: $Assoc(n) = S_n$, with right action of S_n on itself;
$S_n \times S_m \to S_{n+m-1}$, $i \in \{1, ..., n\}$ is given by block insertion. Why is it called $Assoc$? Because an object A in a monoidal category is an associative algebra iff it has the structure of $Assoc$-algebra, i.e. $Hom_{op}(Assoc, End_A) \cong$ monoidal structure on A ($id \otimes A^{\otimes 2} \to A$ gives the multiplication. The other direction is permuting the factor); Also note that the associativity axioms in operads means that insertion of operations are associative, it doesn't mean that the operation is associative in general. The associativity of the operation in this case comes from the fact that the image of (id, id) under $\circ_i : S_2 \times S_2 \to S_2$ is the same for all i, namely id.

If $\mathcal{O}$ is an operad in $(C, \otimes, 1)$, and $F : C \to C'$ is a (lax) monoidal functor, then $F(\mathcal{O})$ is an operad in C'. The associative operad in k-modules is $F(Assoc)$ where F is the free functor, explicitly, the n-th term is $k[S_n]$.

More example:

Operad of parenthesized mutations $Pa : \mathbb{S} \to Set$, e.g. $I = \{1, 2, 3, 4\}$, $(12)(34), ((4(23))1)$ are elements of $Pa(I)$, equivalently $Pa(I)$ is the set of binary rooted trees labelled by I. Note that Pa is free operad generated by $(12) \in Pa(2)$.

Little disk operads: A TD-map is a function $f : D^n \to D^n$ from the open unit disk to itself of the form $ax + b$ where $a > 0$ and $b \in \mathbb{R}^n$.
$E_n(k) := \{(f_1, ...f_k) : f_i TD, im(f_i) \text{ pairwise disjoint}\}$, this can be topologized as a subspace of $(\mathbb{R}_{>0} \times \mathbb{R}^n)^k$.

Commutative operad: $Com(n) = \{*\} = (S_n)_{S_n}$. Structure maps are forced by definition. There is a map of operad $Assoc \to Com$, so if A is a Com-algebra, it is also a Assoc-algebra. There exists map of operads $Pa \to Assoc$ by forgetting the parentheses. What is a Pa-algebra, it is just binary operations on X.

In $Vect_k$, recall $Assoc(n) = k[S_n]$, then the Lie operad is the suboperad generated by $[-, -] := id - (12) \in Assoc(2)$. Jacobi idenitty is forced by the suboperad. We will see that Lie is freely generated by $Lie(2) = k[-, -]$ with the sign representation of S_2 subject to the Jacobi identity.

Free operads: Left adjoint to the forgetful functor from operads to symmetric sequences

Graph:= collection of half-edges (connected to one vertex), a vertex is an equivalence class of half edges, an involution on the half edges, fixed points are legs, internal edges are pairs of half edges swapped under involution

We can produce a 1D CW complex homeomorphic to a graph. A tree is a graph whose top space is simply connected, and rooted tree is a tree with a leg picked as a root, the other legs are leaves. Since the graph is a 1-dimensional simply connected there exists unique path from any vertex from leaf to the root, and we direct the edge according to it. for any vertex v, we have the incoming half edges $I(T)$ and the outgoing half edges.

Digression: how to start from $\mathcal{O}(n)$ each one with an action of S_n to get a functor $\mathcal{O} : \mathbb{S} \to \mathcal{C}$: $\mathcal{O}(I) = (\sqcup_{\rho:[n]\to I}\mathcal{O}(n))_{S_n}$, and this is natural in I (by postcomposing with $I \to J$)

Explicit description of free operad: Let $\mathcal{E}$ be a symmetric sequence: Define $F(\mathcal{E})(I)$ for some finite set $I = \{i_1, ...i_n\}$. The idea is direct summing over all trees and all labels on leaves by I of a certain object in $\mathcal{C}$. The object is

$$T\langle\mathcal{E}\rangle = \otimes_{v\in V(T)}\mathcal{E}(I_n(V))$$

and we can define $F(\mathcal{E})(I) = \varinjlim_T T\langle\mathcal{E}\rangle$

Ideals of operads: collection of subobject $I(n) \subset \mathcal{O}(n)$ closed under insertion from left. A presentation of an operad is an isomorphism $Free(P)/I \to \mathcal{O}$

Examples: In Set, Pa is the free operad on Q where Q is a symmetric sequence with S_2 at degree 2 and $\emptyset$ otherwise. In vec, $Assoc$ is $Pa/(\mu \circ_1 \mu - \mu \circ_2 \mu)$ where μ is the nonidentity element of S_2 (see Ginzburg-Kapranov paper for reference). Similarly for the Lie operad (Intuition for why Hall words give a graded basis for free Lie algebra.)