Monoidal categories and monoidal functors

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Intuition for monoidal categories: Every (small) monoidal category may also be viewed as a “categorification” of an underlying monoid, namely the monoid whose elements are the isomorphism classes of the category’s objects and whose binary operation is given by the category’s tensor product.

Braided monoidal categories: natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ satisfying the hexagon axiom. symmetric monoidal categories: $\sigma A, B^{-1} = \sigma_{A,B}$. Example: product of two sets, tensor product of vector spaces, graded vector space with braiding given by Koszul sign convention

An algebra in a monoidal category is a monoidal object A (bilinearity is distributive law). Similarly we can define coalgebras. If $\mathcal{C}$ is braided and A, B are algebras, then so is $A \otimes B$. A bialgebra in a braided category is a algebra and coalgebra s.t. comultiplication is a map of algebras. It is moreover a Hopf algebra if there is an antipode (like inverse) $S : A \to A$ such that $A \to A \otimes A \xrightarrow{S \otimes 1} A \otimes A \to A$ is equal to the composite $A \to 1 \to A$.

Monoidal functor (lax): $F : C \to D$ is monoidal if there exists natural transformation $F(A) \otimes F(B) \to F(A \otimes B)$ (colax if it is the other direction)

If A is an algebra in C and F is lax monidal then $F(A)$ is an algebra in D (coalgebra if it is colax monoidal)

A very nontrivial class of braided monoidal cateogies is that of Yetter-Drinfeld modules: G group, yD_G^G be the category with object right $k[G]$-modules which decompose $V = \oplus_{g \in G} V_g$ such that $V_g \cdot h = V_{g^h}$ where $g^h = h^{-1} g h$ and morphisms are linear maps preserving the action and the grading. The monoidal structure is $(V \otimes W)_g = \oplus_{g_1 g_2 = g} V_{g_1} \otimes W_{g_2}$ with action via diagonal action. The braiding $\sigma : V \otimes W \to W \otimes V$ is going to send $V_{g_1} \otimes W_{g_2}$ to $W_{g_2} \otimes V_{g_1^{g_2}}$ (similar to semidirect product) where $v \otimes w \mapsto w \otimes (v \cdot g_2)$.