automorphy lifting: Lecture 10-13

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If the characteristic of the coefficient field is the same as that of the residual characteristic of the local field, we need tools from p-adic Hodge theory.

Cohen structure theorem: Suppose $(R, \mathfrak{m})$ is equal characteristic CDVR (complete discrete valuation ring). Then the canonical ring homomorphism $R \to k := R/\mathfrak{m}$ splits ring theoretically, i.e. there exists subring $R_0 \to R$ such that $R_0 \to R \to k$ is an isomorphism, and any choice of $t \in \mathfrak{m} \setminus \mathfrak{m}^2$ induces $R \cong k[[t]]$. When R is mixed characteristic $(0, p)$, in this case, there is a coefficient subring R_0 with the following property: 1. R_0 is a CDVR with residue field isomorphic to k via $R \to k$. 2. R_0 is absolutely unramified, i.e. unramified over $\mathbb{Z}_p$, or $\mathfrak{m} = (p)$. This R_0 is determined up to isomorphism by k, but not uniquely/functorially. Such a coefficient ring is also called Cohen p-ring.

If k is not perfect, then R_0 can be very noncanonical. For example, if we take $k = \mathbb{F}_p(t)$, and $R = \widehat{\mathbb{Z}_p[t]_{(p)}}$. Then R is a CDVR with unique maximal ideal $\mathfrak{m} = (p)$ and $R/\mathfrak{m} = k$. But $t \mapsto t + p$ is an automorphism of R that reduced to identity mod $\mathfrak{m}$.

If k is perfect, there is a functor of Witt vectors from the category of perfect fields of characteristic p to that of Cohen p-rings, splitting the functor of taking reduction mod p, and it is unique up to unique isomorphism. The functor uniquely extends to a functor from the category of perfect (Frobnius map is an isomorphism) $\mathbb{F}_p$-algebras to that of p-adically complete ($A \cong \varprojlim A/p^r A$) flat (torsion-free) $\mathbb{Z}_p$-algebras with A/pA perfect.

Example: $W(\mathbb{F}_{p^r}) = \mathbb{Z}_p(\zeta_{p^r-1})$, $W(\overline{\mathbb{F}_p}) \cong \mathcal{O}_{\mathbb{Q}_p^{ur}}$.

p-adic expansions: If $x, y \in A$, $x - y \in p^r A$ for some $r \geq 1$, then $x^p - y^p \in p^{r+1}A$ (expand $(x-y)^p$). There is a unique map $[] : A/pA \to A$ preserving multiplication such that $[x] \mod p = x$ and $[x]$ has p^n-th root for all $n \geq 1$.

The idea is to construct a Cauchy sequence that reduces to x. Take $y_n \in A/pA$ such that $y_n^{p^n} = x$ (which exists since A/pA is perfect). Pick any lift $\tilde{y_n} \in A$, then $\tilde{y_n}^{p^n} - \tilde{y_{n+1}}^{p^{n+1}} \in p^{n+1}A$. By completeness of A, $\{\tilde{y_n}^{p^n}\}$ converges.

Since A is flat, we can divide by p and get a p-adic expansion $x = [x_0] + p[x_1^{1/p}] + p^2[x_2^{1/p}] + ...$

The question is that if we can write down $z = x + y$ using x_i and y_i. First $z_0 = x + y \mod p = x_0 + y_0$. Then $z_1^{1/p} = \frac{x+y-[z_0]}{p} \mod p = ([x_1^{1/p}] + [y_1^{1/p}] + \frac{[x_0]+[y_0]-[x_0+y_0]}{p}) \mod p$. Raising both sides to the p-th power, then we get $z_1 = x_1 + y_1 + \frac{x_0^p+y_0^p-(x_0+y_0)^p}{p}$. So on and so forth.

For multiplication $w = xy$, we similarly expand and find out $w_1 = x_0^p y_1 + x_1 y_0^p$. In general, given $\Phi(u, v) \in \mathbb{Z}[u, v]$, we can find $\Phi(x, y) = t = \sum_{i=0}^{\infty} p^i [t_i]^{1/p^i}$ such that $t_i = \Phi_i(x_0, ...x_i, y_0, ...y_i)$ is a polynomial over $\mathbb{Z}$ independent of A.

The idea is to we can use the representative $[0, ..., p - 1]$ as coefficient, but they do not have good properties (not closed under addition or multiplication), while the Teichmuller representatives are closed under multiplication.

We now turn the process around and use the universal polynomials to construct $W(R)$ from R.

Important maps: Teichmuller $[] : R \to W(R)$ mapping z to $(z, 0, 0, ...)$ which is multiplicative. Then for $x = (x_0, x_1, ...)$, we can write it as $\sum_{i=0}^{\infty} p^i [x_i^{1/p^i}]$. Since $p(x_0, ...) = \sum_{i=0}^{\infty} p^{i+1} [x_i^{1/p^i}] = (0, x_0^p, ...)$, we see that p is not a zero divisor. Frobenius map $F : W(R) \to W(R)$ given by $(x_0, x_1, ...) \mapsto (x_0^p, x_1^p, ...)$ and Verschiebung $V : W(R) \to W(R)$ given by $(x_0, ...) \mapsto (0, x_0, ...)$ and by definition we have $F \circ V = V \circ F = p$.

We define $W_n(R) := W(R)/p^n W(R)$ (keeping only the first n-coordinates) and we have $W(R) \cong \varprojlim W_n(R)$ which implies $W(R)$ is p-adic complete and its reduction mod p is R. Since the universal polynomials Φ_i^+, Φ_i^* is independent in R, the Witt vector construction is functorial in R. In particular, $W(R)$ is canonically a $\mathbb{Z}_p$-algebra. Since p is not zero divisor in $W(R)$, it is a flat $\mathbb{Z}_p$-algebra.

If B is a p-adic complete flat $\mathbb{Z}_p$-algebra with perfect B/pB, then any $\mathbb{F}_p$-homomorphism $f : A/pA \to B/pB$ uniquely lifts to $\tilde{f} : A \to B$ given by $\sum_{i=0}^{\infty} p^i [x_i^{1/p^i}] \mapsto \sum_{i=0}^{\infty} p^i [f(x_i^{1/p^i})]$. Note that even when $[w]$ is not defined for all elements $w \in B/pB$ (e.g. B/pB is not perfect), it is defined for $w \in f(A/pA)$ for $a \in A$. Moreover, if B is flat over $\mathbb{Z}_p$ or $\mathbb{Z}/p^n\mathbb{Z}$ for some n (implying $p^i B/p^{i+1} B \cong B/pB$ or 0), then the same universal polynomial for $+$ and $\cdot$ can still make sense in B. Thus $\tilde{f} : A \to B$ is still a ring homomorphism as long as B is a p-adic complete $\mathbb{Z}_p$-algebra and flat over $\mathbb{Z}_p$ or $\mathbb{Z}/p^n$ for some n. However, note that $W(B/pB)$ is not isomorphic to B since it is not flat (see criterion below).

For perfect $\mathbb{F}_p$-algebra R, $W(R)$ can be thought of as the **unique** deformation of R to a p-adic complete flat $\mathbb{Z}_p$-aglebra. The argument is based on deformation theory, using $\mathbb{L}_{R/\mathbb{F}_p} = 0$ (essentially boils down to the derivative of $x \mapsto x^p$ is zero).

Remark: There is an explicit creiterion that R is perfect iff $W_2(R)$ flat over $\mathbb{Z}/p^2$ (since the flatness is equivalent to $0 \to R \to W_2(R) \to R \to 0$ being exact, where the first map given by $x \mapsto (0, x^p)$ and the second one given by $(x_0, x_1) \mapsto x_0$).

Hodge-Tate and de Rham's representation: Let $C = \widehat{\overline{F_v}} = \widehat{\overline{\mathbb{Q}_p}}$ and $\mathcal{O}_C$ and $\mathfrak{m}_C$ similarly. We have $\mathcal{O}_C/\mathfrak{m}_C \cong \overline{k_v}$. It is certainly not discretely valued. Note that we have $\mathfrak{m}_{\mathcal{O}_C}^2 = \mathfrak{m}_{\mathcal{O}_C}$ (allows the use of almost mathematics).

The ring C is an example of perfectoid field (a complete topological field K with topology induced by a non-discrete valuation such that the arithmetic Frobenius $\Phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective). Here is a quick summary of some results on perfectoid fields, for details see Scholze's original paper.

A perfectoid field of characteristic p is the same as a complete perfect nonarchimedean field. The non-discrete valuation condition guarantees that the value group is p-divisible. It is related to the notion of deeply ramified fields. Next we describe the process of tilting for perfectoid fields, which is a functor that takes as input a perfectoid field and produce a perfectoid field in characteristic p.

Choose any $\varpi \in K^\times$ such that $|p| \leq |\varpi| < 1$. Define

$$\mathcal{O}_{K^\flat} := \varprojlim_{\Phi} \mathcal{O}_K/\varpi$$

where Φ is the Frobenius morphism (note that $\mathcal{O}_K/\varpi$ is a highly nonreduced $\mathbb{F}_p$-algebra and by taking inverse limit we have made $\mathcal{O}_{K^\flat}$ into a perfect ring of characteristic p). Equip it with the inverse limit topology where each $\mathcal{O}_K/\varpi$ is given the discrete topology. We first claim there is a map $\# : \mathcal{O}_{K^\flat} \to \mathcal{O}_K$. This map is similar to the construction of Teichmuller representative (this just uses that $\varpi \mid p$). So similar to $[]$, $\#$ is multiplicative and continuous. Using $\#$ we can further define $\mathcal{O}_{K^\flat} \to \varprojlim_{x \mapsto x^p} \mathcal{O}_K$ by $x \mapsto (x^\#, (x^{1/p})^\#, ...)$ which is inverse to the projection map. This shows that the two inverse limits are isomorphic as topological multiplicative monoid.

Secondly, There is an element $\varpi^\flat \in \mathcal{O}_{K^\flat}$ such that $|(\varpi^\flat)^\#| = |\varpi|$ (pick any $\varpi_1 \in K$ with $|\varpi_1|^p = |\varpi|$ and choose any sequence $\varpi^\flat = (0, \varpi_1, ...) \in \mathcal{O}_{K^\flat}$). If we define $K^\flat = \mathcal{O}_{K^\flat}[1/\varpi^\flat]$, then $\#$ extends to $K^\flat \to \varprojlim K$ (note that it is harmless to replace ϖ by $(\varpi^\flat)^\#$), which is easily seen to be a homeomorphism. In particular $K^\flat$ is a field. The topology on $K^\flat$ is induced $x \mapsto |x^\#|$.

Fact: Finite extension of perfectoid field is perfectoid and the tilting functor $L \mapsto L^\flat$ defines an equivalence of category between finite extensions of K and finite extensions (Theorem 3.7), the proof of which uses almost mathematics, the key being the following string of equivalences of categories:

$$K_{fet} \cong (O_K^a)_{fet} \cong (\mathcal{O}_K^a/\varpi)_{fet} = (\mathcal{O}_{K^\flat}^a/\varpi^\flat)_{fet} \cong (\mathcal{O}_{K^\flat}^a/\varpi)_{fet} \cong K_{fet}^\flat.$$

In particular, $G_K \cong G_{K^\flat}$. For example, if we take $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, then $K^\flat = \widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$. The reduction $\mathcal{O}_K/p$ is isomorphic (multiplicatively) to $\mathcal{O}_{K^\flat}$ mod t. Note that the absolute Galois group of $K^\flat$ is just $G_{\mathbb{F}_p((t))}$ since taking p^n-th root of t gives purely inseparable extensions.

We are interested in G_{F_v} representations of the form $W := \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$, but since G_{F_v} acts on $\mathbb{C}_p$ (since $v \mid p$), it is not a $\mathbb{C}_p$-linear representation, but rather a semi-linear representation. There is a standard recipe to build semi-linear representations, namely if $G \to GL_E(V)$ is a ordinary linear representation and B is an E-algebra such that G acts on B (e.g. $B = L$ and $G = Gal(L/E)$) then $B \otimes_E V$ is a semi-linear representation. In particular, if $\chi : G \to E^\times$ is a character, then $B(\chi) = Be_\chi$ is a semilinear representation defined by $g \cdot e\chi = \chi(g)e_\chi$.

We call a B-semilinear representation is trivial if it is isomorphic to B^n for some n. Note that B-semi-linear representation of G is trivial if and only if it admits a basis of vectors which are fixed by G. In particular, it is quite possible that a nontrivial semi-linear representation becomes trivial after scalar extension. Given $W \in Rep_B(G)$, we denote by W^G the subset of W consisting of fixed points under G, Clearly W^G is a module over B^G. Moreover scalar extension provides a canonical morphism in $Rep_B(G)$:

$$\alpha_W : B \otimes_{B^G} W^G \to W.$$

This is useful for recognizing trivial semi-linar representations since if W is trivial, α_W will be an isomorphism by virtue of $(B^n)^G \cong (B^G)^n$. The converse holds when W and W^G are free of finite rank over B and B^G respectively (the intuition is that we can detect a trivial B-semi-linear representation arise from a trivial B^G-linear representation).

If G is finite acting on a field L, then L/L^G is a finite Galois extension with Galois group G. Hilbert 90 can be reformulated by saying that $\alpha_W : L \otimes_{L^G} W \to W$ is always surjective, and if W is finite-dimensional, then α_W is bijective, W is trivial semi-linear representation, see this survey paper, Theorem 1.3.3 for details. Note that this fails if L/L^G is an infinite extension, e.g. $\overline{\mathbb{Q}_p} \ncong \overline{\mathbb{Q}_p}(\chi_{cycl})$, since there is no $x \in \overline{\mathbb{Q}_p}$ such that $gx = \chi(g)x$ for every $g \in G_{\mathbb{Q}_p}$, see Example 1.3.5 for detail.

Now we come to an important definition, a finite-dimensional representation $V \in Rep_E(G)$ is B-admissible if $B \otimes_E V$ is trivial. A numerical criterion for recognizing B-admissible representations is that $\dim_{B^G} W^G = \dim_E V = \dim_B B \otimes_E V$, provided B satisfies some requirements (Proposition 1.4.4).

Fact: Let V be a $\mathbb{Q}_p$-linear finite dimensional representation of G_K. Then V is $\mathbb{C}_p$-admissible if and only if the inertia subgroup of G_K acts on V through a finite quotient (Theorem 1.4.6). Thus, $\mathbb{C}_p$-admissibility detects those representations which are potentially unramified. In particular, the cyclotomic character χ_{cycl} are not $\mathbb{C}_p$-admissible. Then V is Hodge-Tate iff it is B_{HT}-admissible.

A larger class of G_K-representations: A $\mathbb{Q}_p$-linear finite-dimensional representation V of G_K is Hodge-Tate if

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \mathbb{C}_p(\chi_{cycl}^{n_1}) \oplus ... \oplus \mathbb{C}_p(\chi_{cycl}^{n_d})$$

. This fits into the framework of B-admissibility as follows: Let $B_{HT} = \mathbb{C}_p[t, t^{-1}]$ and $g \cdot t^i = \chi_{cycl}(g)^i t^i$.

Theorem 1.4.6 is the starting point for studying Hodge–Tate representations. For example, it implies that the integers n_i's that appeared above are uniquely determined up to permutation (Proposition 2.2.8). They are called the Hodge–Tate weights of the representation V. Finally, Hodge-like decomposition theorems show that many representations coming from geometry are Hodge–Tate.

Unfortunately, Hodge–Tate representations have several defaults. First, they are actually too numerous and, for this reason, it is difficult to describe them precisely and design tools to work with them efficiently. The second defect of Hodge–Tate representations is of geometric nature. Indeed, tensoring the etale cohomology with $\mathbb{C}_p$ (or equivalently, with B_{HT}) captures the graded module of the de Rham cohomology. However, it does not capture the entire complexity of de Rham cohomology, the point being that the de Rham filtration does not split canonically in the p-adic setting.

In order to work around this issues, Fontaine defined other period rings B 'finer' than B_{HT}. The most classical period rings introduced by Fontaine are $B_{crys} \subset B_{st} \subset B_{dR}$; the corresponding admissible representations are called crystalline, semi-stable and de Rham respectively. Moreover, B_{dR} is a filtered field whose graded ring can be canonically identified with B_{HT}. This property, together with the aforementionned inclusions, imply the following implications (since if $B_1 \subset B_2$, or B_2 is an algebra over B_1 and V is B_1-admissible, then it is B_2 admissible): crystalline implies semi-stable implies de Rham implies Hodge–Tate.

Rapidly, let us say here that representations coming from the geometry, i.e. of the form $H^r_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ where X is a smooth projective algebraic variety over $\mathbb{Q}_p$, are all de Rham. By definition, this means that the space $(B_{dR} \otimes_{\mathbb{Q}_p} H^r_{et}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K}$ has the correct dimension. It turns out that this space has a very pleasant cohomological interpretation: it is canonically isomorphic to the de Rham cohomology of X, namely $H^r_{dR}(X)$. We thus get an isomorphism:

$$B_{dR} \otimes_{\mathbb{Q}_p} H^r_{et}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{dR} \otimes_K H^r_{dR}(X)$$

The introduction of B_{dR} resolves elegantly the geometric issue we have pointed out earlier. However, the class of B_{dR}-admissible representations is still rather large and not easy to describe. The ring B_{crys} is a subring of B_{dR} which is equipped with more structures and provides very powerful tools for describing crystalline representations. On the geometric side, crystalline representations correspond to the etale cohomology of varieties with good reduction and the space $(B_{crys} \otimes_{\mathbb{Q}_p} H^r_{et}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K}$ is related to the crystalline cohomology of (the special fibre of a proper smooth model of X, equipped with its Frobenius endomorphism.

Let $A_{inf} := W(\mathcal{O}_{C^\flat})$ and $B^+_{inf} := A_{inf}[1/p]$. It has a strong geometrical interpretation observed first by Colmez and then by Fargues–Fontaine and Scholze that B^+_{inf} appears at a mixed characteristic analogue of the ring of bounded analytic functions on the open unit disc. There is an important map $\theta : A_{inf} \to \mathcal{O}_C$ lifting $\mathcal{O}_{C^\flat} \to \mathcal{O}_{C^\flat}/\varpi^\flat \cong \mathcal{O}_C/\varpi$. Concretely, it is given by

$$\theta : \sum_{i=0}^{\infty} [\xi_i] p^i \mapsto \sum_{i=0}^{\infty} \xi_i^{\#} p^i.$$

The kernel $ker\theta$ is a principal ideal generated by $\xi := [p^\flat] - p$. It turns out the element $\omega := \frac{[\epsilon]-1}{[\epsilon^{1/p}]-1}$ also generates $ker(\theta)$, where ϵ is a compatible system of p-th power roots of unity.

We now define $B^+_{dR} = \varprojlim_n B^+_{inf}/ker(\theta)^n$ and $B_{dR} = Frac(B^+_{dR}) = B^+_{dR}[1/\xi]$. The key feature of B^+_{dR} is that it's a CDVR with residue field $\mathbb{C}_p$ (see Tony Feng's thesis, Prop. 8.18). We equip B^+_{dR} with the topology from B^+_{inf} (so that it induce the usual topology on $\mathbb{C}_p$. There is a special element in B_{dR} that is a period for the cyclotomic character, i.e. G_K acts by multiplication by χ_{cycl}, namely $t := \log([\epsilon])$. Note that the $\mathbb{Z}_p$-line generated by t is independent of the choice of ϵ, which can be thought of as analogous to $2\pi i\mathbb{Z}$ in complex analysis, and the element t as analogous to a choice of $2\pi i$.

Fact: t is a uniformizer for B^+_{dR} and thus the associated graded algebra of B_{dR} is isomorphic (Galois-equivariantly) to B_{HT}.

One idea to create B_{dR} is that we want to functorially build a complete discrete valuation ring with residue field $\mathbb{C}_p$ of characteristic 0. Naturally the Witt vector construction comes to mind, but we need to be more artful here since we are in the equicharacteristic zero situation. Note that any complete discrete valuation ring with residue field F of characteristic 0 is abstractly isomorphic to $F[[t]]$ by commutative algebra, such a structure will not exist for B_{dR}^+ in a G_K-equivariant manner. Rather than trying to directly make a canonical complete discrete valuation ring with residue field $\mathbb{C}_K$, we observe that $\mathbb{C}_K = \mathcal{O}_{\mathbb{C}_K}[1/p]$ which is closely related to p-power torsion rings. Hence, it is more promising to try to adapt Witt-style constructions for $\mathcal{O}_{\mathbb{C}_K}$ than for $\mathbb{C}_K$. We will make a certain height-1 valuation ring R of equicharacteristic p whose fraction field $Frac(R)$ is algebraically closed (hence perfect) such that there is a natural G_K-action on R and a natural surjective G_K-equivariant map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$. (Note that $W(R) \subset W(Frac(R))$, so $W(R)$ is a domain of characteristic 0.) We would then get a surjective G_K-equivariant map $\theta_{\mathbb{Q}} : W(R)[1/p] \to \mathcal{O}_{\mathbb{C}_K}[1/p] = \mathbb{C}_K$. Since R is like a 1-dimensional ring, $W(R)$ is like a 2-dimensional ring and so $W(R)[1/p]$ is like a 1-dimensional ring. The ring structure of $W(A)$ is generally pretty bad if A is not a perfect field of characteristic p, but as long as the maximal ideal $ker(\theta_{\mathbb{Q}})$ is principal and nonzero we can replace $W(R)[1/p]$ with its $ker(\theta_{\mathbb{Q}})$-adic completion to obtain a canonical complete discrete valuation ring B_{dR}^+ having residue field $\mathbb{C}_K$.

Reference: https://math.stanford.edu/~conrad/papers/notes.pdf