automorphy lifting: Lecture 10-13

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If the characteristic of the coefficient field is the same as that of the residual characteristic of the local field, we need tools from p-adic Hodge theory.

Cohen structure theorem: Suppose $(R, \mathfrak{m})$ is equal characteristic CDVR (complete discrete valuation ring). Then the canonical ring homomorphism $R \to k := R/\mathfrak{m}$ splits ring theoretically, i.e. there exists subring $R_0 \to R$ such that $R_0 \to R \to k$ is an isomorphism, and any choice of $t \in \mathfrak{m} \setminus \mathfrak{m}^2$ induces $R \cong k[[t]]$. When R is mixed characteristic $(0, p)$, in this case, there is a coefficient subring R_0 with the following property: 1. R_0 is a CDVR with residue field isomorphic to k via $R \to k$. 2. R_0 is absolutely unramified, i.e. unramified over $\mathbb{Z}_p$, or $\mathfrak{m} = (p)$. This R_0 is determined up to isomorphism by k, but not uniquely/functorially. Such a coefficient ring is also called Cohen p-ring.

If k is not perfect, then R_0 can be very noncanonical. For example, if we take $k = \mathbb{F}_p(t)$, and $R = \widehat{\mathbb{Z}_p[t]_{(p)}}$. Then R is a CDVR with unique maximal ideal $\mathfrak{m} = (p)$ and $R/\mathfrak{m} = k$. But $t \mapsto t + p$ is an automorphism of R that reduced to identity mod $\mathfrak{m}$.

If k is perfect, there is a functor of Witt vectors from the category of perfect fields of characteristic p to that of Cohen p-rings, splitting the functor of taking reduction mod p, and it is unique up to unique isomorphism. The functor uniquely extends to a functor from the category of perfect (Frobnius map is an isomorphism) $\mathbb{F}_p$-algebras to that of p-adically complete ($A \cong \varprojlim A/p^r A$) flat (torsion-free) $\mathbb{Z}_p$-algebras with A/pA perfect.

Example: $W(\mathbb{F}_{p^r}) = \mathbb{Z}_p(\zeta_{p^r-1})$, $W(\overline{\mathbb{F}_p}) \cong \mathcal{O}_{\mathbb{Q}_p^{ur}}$.

p-adic expansions: If $x, y \in A$, $x - y \in p^r A$ for some $r \geq 1$, then $x^p - y^p \in p^{r+1} A$ (expand $(x-y)^p$). There is a unique map $[] : A/pA \to A$ preserving multiplication such that $[x] \mod p = x$ and $[x]$ has p^n-th root for all $n \geq 1$.

The idea is to construct a Cauchy sequence that reduces to x. Take $y_n \in A/pA$ such that $y_n^{p^n} = x$ (which exists since A/pA is perfect). Pick any lift $\tilde{y_n} \in A$, then $\tilde{y_n}^{p^n} - \tilde{y_{n+1}}^{p^{n+1}} \in p^{n+1} A$. By completeness of A, $\{\tilde{y_n}^{p^n}\}$ converges.

Since A is flat, we can divide by p and get a p-adic expansion $x = [x_0] + p[x_1^{1/p}] + p^2[x_2^{1/p}] + \ldots.$

The question is that if we can write down $z = x + y$ using x_i and y_i. First $z_0 = x + y \mod p = x_0 + y_0$. Then $z_1^{1/p} = \frac{x+y-[z_0]}{p} \mod p = ([x_1^{1/p}] + [y_1^{1/p}] + \frac{[x_0]+[y_0]-[x_0+y_0]}{p}) \mod p$. Raising both sides to the p-th power, then we get $z_1 = x_1 + y_1 + \frac{x_0^p+y_0^p-(x_0+y_0)^p}{p}$. So on and so forth.

For multiplication $w = xy$, we similarly expand and find out $w_1 = x_0^p y_1 + x_1 y_0^p$. In general, given $\Phi(u, v) \in \mathbb{Z}[u, v]$, we can find $\Phi(x, y) = t = \sum_{i=0}^{\infty} p^i [t_i]^{1/p^i}$ such that $t_i = \Phi_i(x_0, ...x_i, y_0, ...y_i)$ is a polynomial over $\mathbb{Z}$ independent of A.